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A Note on the Ratio of the Consecutive Coefficients of an Analytic Function Represented by Dirichlet Series

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ABSTRACT

Recently [Indian J. Pure Appl. Math. 10 (1979), 171-182] we derived formula for the logarithmic order of an analytic function of zero order represented by Dirichlet series in terms of the coefficients. In the present paper we obtain inequalities for the logarithmic order in terms of the ratio of the consecutive coefficients. These inequalities are shown to be the best possible.

INTRODUCTION

1. Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \tag{1.1}$$

where $0 \le \lambda_1 < \lambda_2 < \ldots < \lambda_n \ldots, \lambda_n \to \infty$ as $n \to \infty$, $s = \sigma + it$ (σ, t being real variables), $\{a_n\}_1^{\infty}$ is a sequence of complex numbers and

$$\limsup_{n \to \infty} \left(\frac{n}{\lambda_n}\right) = D < \infty . \tag{1.2}$$

If the series given by (1.1) converges absolutely in the halfplane Re s $< \alpha$ ($-\infty < \alpha < \infty$) then it is known [4,p.166] that the series (1.1) represents an analytic function in Re s $< \alpha$ and since (1.2) is satisfied we have

$$\alpha = -\limsup_{n \to \infty} \frac{\log |a_n|}{\lambda_n} .$$

Set,

$$M(\sigma,f) = M(\sigma) = \underset{-\infty<t<\infty}{\text{l. u. b.}} |f(\sigma + it)|$$

$$m(\sigma, f) = m(\sigma) = \max_{n\geq 1} (|a_n| e^{\sigma\lambda_n})$$

and

$$N(\sigma) = \max \{n: m(\sigma) = |a_n| e^{\sigma\lambda_n}\}.$$

It is known [4] that log M (σ) is an increasing convex function of σ for $\sigma < \alpha$. For a function f (s) analytic in the half-plane Re s $< \alpha$ $(-\infty < \alpha < \infty)$, Krishna Nandan [3] has defined the order ρ and lower order λ $(0 \leq \lambda, \rho \leq \infty)$ as

$$\lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M(\sigma)}{-\log \{1-\exp(\sigma-\alpha)\}} = \begin{matrix} \rho \\ \lambda \end{matrix}.$$

The above growth parameters do not give any specific information about the growth of f (s) when $\rho = 0$. Recently we [1] have studied the growth of such functions by comparing the growth of log log $M(\sigma)$ with that of log log $\{1\text{-exp}(\sigma-\alpha)\}^{-1}$ and obtained formula for logarithmic order ρ^* in terms of the coefficients a_n's and exponents λ_n's.

Thus if D^*_α denote the class of all functions f (s) of zero order, which are analytic in the half-plane Re s $< \alpha$ and are defined by (1.1), and that f (s) $\in A^*_\alpha \subset D^*_\alpha$ if and only if $m(\sigma) / \{1\text{-exp}(\sigma-\alpha)\}^{-\varepsilon} \to \infty$ as $\sigma \to \alpha$ for some $\varepsilon > 0$, then the logarithmic order ρ^* and lower logarithmic order λ^* of f (s) $\in A^*_\alpha$ are defined [1] as

$$\lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M(\sigma)}{\log\log \{1-e^{\sigma-\alpha}\}^{-1}} = \begin{matrix} \rho^* \\ \lambda^* \end{matrix} \quad (1 \leq \lambda^*, \rho^* \leq \infty),$$

and it is shown [1]

$$\rho^* = \max \{1,\theta\}$$

where $\theta = \limsup_{n \to \infty} \dfrac{\log^+(\alpha\lambda_n + \log |a_n|)}{\log\log \lambda_n}$.

Our main aim in the present paper is to find inequality for logarithmic order ρ^* in terms of the ratio of the consecutive coefficients and we show that these inequalities are best possible.

Theorem. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *belong to class* A^*_α *having logarithmic order* ρ^* $(1 \leq \rho^* \leq \infty)$. *If*

$\psi(n) = \dfrac{\log |a_n/a_{n+1}|}{\lambda_{n+1} - \lambda_n} \geq \alpha - 1$ *for all* $n > n_0$, *and is a non-decreasing function of* n *for* $n > n_0$, *then*

$$\rho^* - 1 \leq \limsup_{n \to \infty} \frac{\log^+ \left[\lambda_n \log^+ \left\{ (\exp \alpha) \left| \frac{a_n}{a_{n-1}} \right|^{\frac{1}{\lambda_n - \lambda_{n-1}}} \right\} \right]}{\log \log \lambda_n} \leq \rho^* \quad (2.1)$$

Proof. Since $\psi(n) \geq \alpha - 1$ for all $n > n_0$, we have $0 \leq \theta \leq \infty$, where

$$\theta = \limsup_{n \to \infty} \frac{\log^+ \left[\lambda_n \log^+ \left\{ (\exp \alpha) \left| \frac{a_n}{a_{n-1}} \right|^{\frac{1}{\lambda_n - \lambda_{n-1}}} \right\} \right]}{\log \log \lambda_n} \quad (2.2)$$

First let $\theta < \infty$. For β such that $\theta < \beta < \infty$, we get for all $n \geq N' = N'(\beta)$,

$$\log \left| \frac{a_n}{a_{n-1}} \right| < (\lambda_n - \lambda_{n-1}) \left[\frac{(\log \lambda_n)^\beta}{\lambda_n} - \alpha \right].$$

Therefore, if $n > N = \max(N', n_0)$, then

$$\begin{aligned} \log |a_n| &= \log |a_N| + \log |a_{N+1}/a_N| + \dots + \log |a_n/a_{n-1}| \\ &< \log |a_N| + (\lambda_{N+1} - \lambda_N) \frac{(\log \lambda_{N+1})^\beta}{\lambda_{N+1}} + \\ &\quad + (\lambda_{N+2} - \lambda_{N+1}) \frac{(\log \lambda_{N+2})^\beta}{\lambda_{N+2}} + \dots \\ &\quad + (\lambda_n - \lambda_{n-1}) \frac{(\log \lambda_n)^\beta}{\lambda_n} - \alpha (\lambda_n - \lambda_N) \\ &= \log |a_N| + (\log \lambda_n)^\beta - \lambda_N \frac{(\log \lambda_{N+1})^\beta}{\lambda_{N+1}} - \alpha (\lambda_n - \lambda_N) \end{aligned}$$

$$- \lambda_{N+1} \left[\frac{(\log \lambda_{N+2})^{\beta}}{\lambda_{N+2}} - \frac{(\log \lambda_{N+1})^{\beta}}{\lambda_{N+1}}\right] - \dots$$

$$- \lambda_{n-1} \left[\frac{(\log \lambda_{n})^{\beta}}{\lambda_{n}} - \frac{(\log \lambda_{n-1})^{\beta}}{\lambda_{n-1}}\right]$$

$$= \log | a_N | + (\log \lambda_n)^{\beta} - \lambda_N \frac{(\log \lambda_{N+1})^{\beta}}{\lambda_{N+1}}$$

$$- \int_{\lambda_{N+1}}^{\lambda_n} n(t)\, d \left[\frac{(\log t)^{\beta}}{t}\right] - \alpha (\lambda_n - \lambda_N), \tag{2.3}$$

where $n(t) = \lambda_m$ for $\lambda_m < t \leq \lambda_{m+1}$; $m = N + 1 \dots n - 1$.

Since

$$\int_{\lambda_{N+1}}^{\lambda_n} n(t)\, d \left[\frac{(\log t)^{\beta}}{t}\right]$$

$$= - \int_{\lambda_{N+1}}^{\lambda_n} n(t) \left[\frac{- \beta (\log t)^{\beta-1} + (\log t)^{\beta}}{t^2}\right] dt$$

$$> - \left[- \frac{(\log t)^{\beta}}{\beta} + \frac{(\log t)^{\beta+1}}{\beta + 1}\right]_{\lambda_{N+1}}^{\lambda_n}$$

$$= - \left[\frac{(\log \lambda_n)^{\beta+1}}{\beta + 1} - (\log \lambda_n)^{\beta}\right]$$

$$+ \left[\frac{(\log \lambda_{N+1})^{\beta}}{\beta + 1} - (\log \lambda_{N+1})^{\beta}\right].$$

Therefore, by (2.3), we have

$$\alpha \lambda_n + \log | a_n | < \log |a_N| - \lambda_N \frac{(\log \lambda_{N+1})^{\beta}}{\lambda_{N+1}} + \alpha \lambda_N$$

$$+ \frac{(\log \lambda_n)^{\beta+1}}{\beta + 1} + (\log \lambda_{N+1})^{\beta} - \frac{(\log \lambda_{N+1})^{\beta+1}}{\beta + 1} \tag{2.4}$$

Hence, for $0 \leq \theta < \beta < \infty$, (2.4) gives

$$\log^{+} (\alpha \lambda_n + \log |a_n|) < (\beta + 1) \log \log \lambda_n + 0 (1).$$

Using (1.3), the last inequality gives $\rho^* \leq \beta + 1$. Since this holds for every $\beta < \theta$, we have $\rho^* - 1 \leq \theta$. If $\theta = \infty$, this inequality is trivially true. Further, since ψ (n) is a nondecreasing function of n for $n > n_0$, for a fixed $N > n_0$ and for all n such that $n > N$, we have

$$\begin{aligned}\log |a_n| &= \log |a_N| + \log \left| \frac{a_{N+1}}{a_N} \right| + \ldots + \log \left| \frac{a_n}{a_{n-1}} \right| \\ &= \log |a_N| - \psi(N)(\lambda_{N+1} - \lambda_N) - \ldots - \psi(n-1)(\lambda_n - \lambda_{n-1}) \\ &\geq \log |a_N| - \psi(n-1)(\lambda_n - \lambda_N).\end{aligned}$$

Hence, by (2.2), for every $\varepsilon > 0$, there exists a sequence of values of n tending to infinity for which

$$\begin{aligned}\alpha \lambda_n + \log |a_n| &> \log |a_N| + (\lambda_n - \lambda_N)\left(\frac{(\log \lambda_n)^{\theta-\varepsilon}}{\lambda_N}\right) + \alpha \lambda_N \\ &= (\log \lambda_n)^{\theta-\varepsilon} \{ 1 + 0(1) \}.\end{aligned}$$

Thus, for infinitely many values of n,

$$\frac{\log^+ \{\alpha \lambda_n + \log |a_n|\}}{\log \log \lambda_n} > \theta - \varepsilon + 0(1). \tag{2.5}$$

Now, passing to limits and using (1.3), we get

$$\rho^* \geq \theta,$$

which is true for $\theta = \infty$ also, since in that case we can take, in place of $(\theta-\varepsilon)$, an arbitrarily large number and then (2.5) gives $\rho^* = \infty$.

Hence the theorem.

Remark. The relation (2.1) is best possible. This can be seen by the following examples.

Example 1. Let $f(s) = \sum_{n=1}^{\infty} \exp [(\log n)^2 + sn]$, then f (s) is analytic in the half-plane Re $s < 0$. Also, by (1.3) $\rho^* = 2$. Now

$$\frac{\log^+ \left[\lambda_n \log^+ \left\{ (\exp \alpha) \left| \frac{a_n}{a_{n-1}} \right|^{\frac{1}{\lambda_n - \lambda_{n-1}}} \right\}\right]}{\log \log \lambda_n}$$

$$= \frac{\log [n \{ (\log n)^2 - (\log (n-1))^2 \}]}{\log \log n}$$

$$\sim \frac{\log (2 \log n)}{\log \log n} \to 1.$$

Thus,

$$\limsup_{n\to\infty} \frac{\log^+\left[\lambda_n \log^+\left\{(\exp \alpha)\left|\dfrac{a_n}{a_{n-1}}\right|^{\frac{1}{\lambda_n-\lambda_{n-1}}}\right\}\right]}{\log\log\lambda_n} = \rho^* - 1,$$

in this case.

Example 2. Let $f(s) = \sum_{n=1}^{\infty} \exp(\log\lambda_n)^2 + s\lambda_n$

where $\log\lambda_1 = 2$, $\log\lambda_{n+1} = (\log\lambda_n)^2$.

Then f (s) is analytic in the half-plane Re s $< \alpha = 0$ by (1,3), $\rho^* = 2$.

Now

$$\frac{\log^+\left[\lambda_n \log^+\left\{(\exp \alpha)\left|\dfrac{a_n}{a_{n-1}}\right|^{\frac{1}{\lambda_n-\lambda_{n-1}}}\right\}\right]}{\log\log\lambda_n}$$

$$\sim \frac{\log\left[\dfrac{\lambda_n\{(\log\lambda_n)^2 - (\log\lambda_{n-1})^2\}}{\lambda_n-\lambda_{n-1}}\right]}{\log\log\lambda_n}$$

$$\frac{\log\ (\log\lambda_n)^2}{\log\log\lambda_n} \to 2.$$

Hence it follows that

$$\limsup_{n\to\infty} \frac{\log^+\left[\lambda_n \log^+\left\{(\exp \alpha)\left|\dfrac{a_n}{a_{n-1}}\right|^{\frac{1}{\lambda_n-\lambda_{n-1}}}\right\}\right]}{\log\log\lambda_n} = 2 = \rho^*,$$

in this case.

Note: It will be of interest to find a function for which strict inequalities hold in (2.1).

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