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The Radius of Starlikeness of Certain Analytic Functions

by

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The Radius of Starlikeness of Certain Analytic Functions

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ABSTRACT

Let P (β) be the class of functions u given by u (z) $= 1+c_1z + c_2z^2+\ldots$ regular in D $= \{z: |z| < 1\}$ and satisfying the condition Re $\{u(z)\} \geq \beta$, $z \in D$, $0 \leq \beta \leq 1$. Let $S^*(\alpha)$ and S (m,M) be the class of functions f given by f (z) $= z+a_2z^2+\ldots$ regular in D and satisfying the conditions

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ z \in D,\ 0 \leq \alpha < 1$$

$$\left|\frac{zf'(z)}{f(z)} - m\right| < M \text{ for } z \in D,\ (m,M) \in E$$

respectively,

where $E = \{(m,M) : |m-1| < M \leq m\}$.

In this paper we obtain the radius of univalence and starlikeness of the set of all functions f that are regular in D and are defined by $f(z)^{l_1} = s(z)^{l_2} u(z)^{l_3} v(z)^{l_4} w(z)^{l_5}$ and $f(z)^{l_1} = s(z)^{l_2} S_1(z)^{l_3} u(z)^{l_4} v(z)^{l_5}$ where $s \in S(m,M)$, $s_1 \in S^*(\alpha)$, $u \in P(\beta)$ or $\frac{1}{u} \in P(\beta)$, $v \in P(\gamma)$ or $\frac{1}{v} \in P(\gamma)$ and $w \in P(\delta)$ or $\frac{1}{w} \in P(\delta)$, l_1, l_2, l_3, l_4, l_5 are all positive real numbers. These results are sharp and include the results of Bhargava [1], Ziegler [7], Causey and Merkes [2] and Ratti [5].

1. INTRODUCTION

Let $S^*(\alpha)$ be the class of functions f given by $f(z) = z + a_2z^2 + \ldots$ regular in $D = \{z: |z| < 1\}$ and satisfying the condition $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$, $z \in D$, $0 \leq \alpha < 1$. Let $P(\beta)$ be the class of functions u given by $u(z) = 1+c_1z+c_2z^2+\ldots$ regular in D and satisfying the condition

Re $\{u(z)\} \geq \beta$, $z \in D$, $0 \leq \beta \leq 1$. Let $S(m,M)$ be the class of functions f given by $f(z) = z+a_2z^2+\ldots$ regular in D and satisfying the condition

$$\left| \frac{zf'(z)}{f(z)} - m \right| < M, \; z \in D, \; (m,M) \in E \text{ where}$$

$$E = \{ (m,M) : |m-1| < M \leq m\}$$

Ziegler [7] obtained the radius of starlikeness of the set all functions f that are regular in D and are of the form

$$(1.1) \qquad f(z) = s(z)\, u(z)\, v(z)$$

where $s \in S^*(\alpha)$, $u \in P(\beta)$ or $\frac{1}{u} \in P(\beta)$, $v \in P(\gamma)$ or $\frac{1}{v} \in P(\gamma)$.

Bhargava [1] obtained the radius of starlikeness of the set of all functions f that are regular in D and are of the form

$$(1.2) \qquad f(z) = s(z)\, u(z)\, v(z)$$

where $s \in S(m,M)$, $u \in P(\beta)$, or $\frac{1}{u} \in P(\beta)$, $v \in P(\gamma)$ or $\frac{1}{v} \in P(\gamma)$.

In this paper we obtain the radius of univalence and starlikeness of the set of all functions which are regular in D and are defined by

$$(1.3) \qquad f(z)^{l_1} = s(z)^{l_2}\, u(z)^{l_3}\, v(z)^{l_4}\, w(z)^{l_5}$$

and

$$(1.4) \qquad f(z)^{l_1} = s(z)^{l_2}\, s_1(z)^{l_3}\, u(z)^{l_4}\, v(z)^{l_5}$$

where $s \in S(m,M)$, $s_1 \in S^*(\alpha)$, $u \in P(\beta)$ or $\frac{1}{u} \in P(\beta)$, $v \in P(\gamma)$ or

$\frac{1}{v} \in P(\gamma)$, $w \in P(\delta)$ or $\frac{1}{w} \in P(\delta)$ and l_1 to l_5 are all positive real numbers.

These results are sharp and generalize the results of Bhargava [1], Ziegler [7], Causey and Merkes [2] and Ratti [5].

2. LEMMAS NEEDED TO PROVE OUR RESULTS

LEMMA 1. If $f \in S^*(\alpha)$ and $|z| \leq r < 1$ then

$$\left| \frac{zf'(z)}{f(z)} \right| \geq \frac{1-(1-2\alpha)\, r}{1+r} = \nu\,(r,\alpha)$$

Equality occurs for the functions $f(z) = \dfrac{z}{(1 \pm z)^{2(1-\alpha)}}$.

LEMMA 2. If $f \in S(m,M)$ and $|z| \leq r < 1$ then

$$\left| \frac{zf'(z)}{f(z)} \right| \geq \frac{1-ar}{1+br} = \mu\,(r,a,b),\ a = \frac{M^2-m^2+m}{M}\ ,b = \frac{m-1}{M}$$

Equality occurs for the functions $f(z) = \dfrac{z}{(1 \pm bz)^{(a+b)/b}}$.

This lemma is due to Silverman [6].

LEMMA 3. If $u \in P(\beta)$ and $|z| \leq r < 1$ then

$$\mathrm{Re} \left\{ \frac{zu'(z)}{u(z)} \right\} \leq \frac{2r(1-\beta)}{(1-r)\ [1+(1-2\beta)r]} = \eta\,(r,\beta)$$

with equality only for $u(z) = \dfrac{1+(1-2\beta)\ \varepsilon z}{1-\varepsilon z}$, $|\varepsilon| = 1.$

This lemma is due to Libera [4].

LEMMA 4. Let β satisfy $0 \leq \beta < 1$. Let $r(\beta)$ denote the root unique on $(2-\sqrt{3},1]$ of the equation

$$(1-2\beta)\ r^3 - 3\,(1-2\beta)\ r^2 + 3r - 1 = 0$$

Let $\sigma_1(r,\beta)$ and $\sigma_2(r,\beta)$ be defined by

$$\sigma_1(r,\beta) = \frac{-2r(1-\beta)}{(1+r)\ [1-(1-2\beta)r\ ]}$$

and

$$\sigma_2(r,\beta) = \frac{-(\sqrt{1+(1-2\beta)r^2} - \sqrt{\beta(1-r^2)}\ )^2}{(1-\beta)\ (1-r^2)},\ 0 \leq r < 1.$$

If $u(z)$ is in $P(\beta)$ then

$$(1.5) \qquad \mathrm{Re} \left\{ \frac{zu'(z)}{u(z)} \right\} \geq \sigma\ (r,\beta) \qquad |z| \leq r < 1$$

where

$$\sigma(r,\beta) = \begin{cases} \sigma_1(r,\beta) & 0 \leq r \leq r(\beta) \\ \sigma_2(r,\beta) & r(\beta) \leq r < 1 \end{cases}$$

if $\beta > 0$ and

(1.6) $$\sigma(r,0) = \sigma_1(r,0),\ 0 \leq r < 1.$$

Equality occurs in (1.5) when $0 \leq r \leq r(\beta)$, only for

$$u(z) = \frac{1+(1-2\beta)\varepsilon z}{1-\varepsilon z},\ |\varepsilon| = 1$$

and when $r(\beta) \leq r < 1$ only for

$$u(z) = \frac{1-2\beta\lambda\ \varepsilon z + (2\beta-1)\varepsilon^2 z^2}{1-2\lambda\ \varepsilon z+\varepsilon^2 z^2},\ |\varepsilon| = 1,$$

where

$$\lambda = \frac{2r}{1+r^2} - \frac{(1-r^2)^{3/2}}{2r(1+r^2)}\left(\frac{1+(1-2\beta)r^2}{\beta}\right)^{1/2}, \quad \beta > 0.$$

Equality occurs in (1.6) only for $u(z) = \dfrac{1+\varepsilon z}{1-\varepsilon z},\ |\varepsilon| = 1.$

This lemma is due to Zmorovic [8].

3. PROOF OF THE FOLLOWING RESULTS

THEOREM 1. Let

$$f(z)^{l_1} = s(z)^{l_2}\ u(z)^{l_3}\ v(z)^{l_4}\ w(z)^{l_5}$$

where $s \in S(m,M)$, $u \in P(\beta)$, $0 \leq \beta < 1$, $v \in P(\gamma)$, $0 \leq \gamma < 1$, $w \in P(\delta)$, $0 \leq \delta < 1$, $\beta \leq \gamma \leq \delta$ and l_1, l_2, l_3, l_4, l_5 are all positive real numbers. Let $F(r,a,b,\beta,\gamma,\delta)$ be defined by

$$(1.7)\quad F(r,a,b,\beta,\gamma,\delta) = \begin{cases} \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\sigma_1(r,\beta) + \frac{l_4}{l_1}\sigma_1(r,\gamma) \\ \quad + \frac{l_5}{l_1}\sigma_1(r,\delta),\ 0 \le r \le r(\delta) \\ \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\sigma_1(r,\beta) + \frac{l_4}{l_1}\sigma_1(r,\gamma) \\ \quad + \frac{l_5}{l_1}\sigma_2(r,\delta),\ r(\delta) \le r \le r(\gamma) \\ \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\sigma_1(r,\beta) + \frac{l_4}{l_1}\sigma_2(r,\gamma) \\ \quad + \frac{l_5}{l_1}\sigma_2(r,\delta),\ r(\gamma) \le r \le r(\beta) \\ \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\sigma_2(r,\beta) + \frac{l_4}{l_1}\sigma_2(r,\gamma) \\ \quad + \frac{l_5}{l_1}\sigma_2(r,\delta),\ r(\beta) \le r < 1 \end{cases}$$

and

$$F(r,a,b,0,0,0) = \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3+l_4+l_5}{l_1}\sigma_1(r,0),\ 0 \le r < 1.$$

Then f is univalent and starlike in $|z| < r_0 < 1$ where r_0 is the smallest positive root of the equation $F(r,a,b,\beta,\gamma,\delta) = 0$. This result is sharp.

Proof:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \frac{l_2}{l_1}\mathrm{Re}\left\{\frac{zs'(z)}{s(z)}\right\} + \frac{l_3}{l_1}\mathrm{Re}\left\{\frac{zu'(z)}{u(z)}\right\}$$

$$+ \frac{l_4}{l_1}\mathrm{Re}\left\{\frac{zv'(z)}{v(z)}\right\} + \frac{l_5}{l_1}\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\}$$

hence by Lemmas 2 and 4

$$(1.8)\quad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \ge \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\sigma(r,\beta) + \frac{l_4}{l_1}\sigma(r,\gamma) + \frac{l_5}{l_1}\sigma(r,\delta)$$

$$= F(r,a,b,\beta,\gamma,\delta), \text{ say.}$$

Let $g(r,\beta) = (1\text{-}2\beta)\, r^3 - 3\,(1-2\beta)\, r^2 + 3\, r - 1$. For each β $(0 \leq \beta < 1)$, $g\,(r,\beta)$ is a strictly increasing function of r, $0 \leq r < 1$ and $g\,(2-\sqrt{3},\beta) = 2\,(1-\beta)\,(5-3\sqrt{3}) < 0$, $g\,(1,\beta) = 4\beta \geq 0$.

Thus $g\,(r,\beta)$ has a unique root $r\,(\beta)$ in $(2-\sqrt{3},1\,]$. Further
$g\,(r(\gamma),\, \beta) = (1-2\beta)\,(r(\gamma)\,)^3 - 3\,(1-2\beta)\,(r(\gamma))^2 + 3r\,(\gamma) - 1$

$$= 2(\gamma-\beta)\,(r(\gamma)\,)^2[\,r(\gamma)-3\,] \leq 0 \text{ if } \beta \leq \gamma$$

hence $r(\gamma) \leq r(\beta)$. Similarly if $\gamma \leq \delta$, $r(\delta) \leq r(\gamma)$. Thus for $\beta \leq \gamma \leq \delta$, $r(\delta) \leq r(\gamma) \leq r(\beta)$ Using these facts F $(r,a,b,\beta,\gamma,\delta)$ defined in (1.8) can be put in the from

(1.7). Since $F(O,a,b,\beta,\gamma,\delta) = \dfrac{l_2}{l_1} > 0$ and $F(r,a,b,\beta,\gamma,\delta) \mapsto -\infty$ as $r \mapsto 1^-$, the equation $F(r,a,b,\beta,\gamma,\delta) = 0$ always has a solution in $(0,1)$.

The sharpness of this result follows from the sharpness of Lemma 2 and 4. In fact, it is possible to choose f(z) so that $\dfrac{r_0 f'(r_0)}{f(r_0)} = F(r_0,a,b,\beta,\gamma,\delta) = 0$ ond thus $f'(r_0) = 0$ i.e. f(z) is not univalent in any disc $|z| < \rho$ if $\rho > r_0$.

THEOREM 2. Let $f(z)^{l_1} = s(z)^{l_2}\, s_1(z)^{l_3}\, u(z)^{l_4}\, v(z)^{l_5}$ where $s \in S(m,M)$, $s_1 \in S^*(\alpha)$, $0 \leq \alpha < 1$, $u \in P\,(\beta)$, $0 \leq \beta \leq 1$, $v \in P\,(\gamma)$, $0 \leq \gamma < 1$, $\beta \leq \gamma$, and l_1, l_2, l_3, l_4, l_5 are all positive real numbers. Let $F(r,a,b,\alpha,\beta,\gamma)$ be defined by

$$F(r,a,b,\alpha,\beta,\gamma) = \begin{cases} \dfrac{l_2}{l_1}\,\mu\,(r,a,b) + \dfrac{l_3}{l_1}\,\nu\,(r,\alpha) + \dfrac{l_4}{l_1}\,\sigma_1(r,\beta) + \dfrac{l_5}{l_1}\,\sigma_1(r,\gamma) & 0 \leq r \leq r(\gamma) \\ \dfrac{l_2}{l_1}\,\mu\,(r,a,b) + \dfrac{l_3}{l_1}\,\nu\,(r,\alpha) + \dfrac{l_4}{l_1}\,\sigma_1(r,\beta) + \dfrac{l_5}{l_1}\,\sigma_2(r,\gamma) & r(\gamma) \leq r \leq r(\beta) \\ \dfrac{l_2}{l_1}\,\mu\,(r,a,b) + \dfrac{l_3}{l_1}\,\nu\,(r,\alpha) + \dfrac{l_4}{l_1}\,\sigma_2(r,\beta) + \dfrac{l_5}{l_1}\,\sigma_2(r,\gamma) & r(\beta) \leq r < 1 \end{cases}$$

and

$$F(r,a,b,\alpha,0,0) = \frac{l_2}{l_1}\,\mu\,(r,a,b) + \frac{l_3}{l_1}\,\nu\,(r,\alpha) + \frac{l_4+l_5}{l_1}\,\sigma_1(r,0),\ 0 \le r < 1$$

Then f is univalent and starlike in $|z| < r_0 < 1$ where r_0 is the smallest positive root of the equation $F(r,a,b,\alpha,\beta,\gamma) = 0$.

This result is sharp.

THEOREM 3. Let $f(z)^{l_1} = \dfrac{s(z)^{l_2}\,u(z)^{l_3}\,v(z)^{l_4}}{w(z)^{l_5}}$ where

$s \in S(m,M)$, $u \in P(\beta)$, $0 \le \beta \le 1$, $v \in P(\gamma)$, $0 \le \gamma \le 1$, $\beta \le \gamma$, $w \in P(\delta)$ $0 \le \delta \le 1$ and l_1, l_2, l_3, l_4, l_5 are all positive real numbers.

Let $F(r,a,b,\beta,\gamma,\delta)$ be defined by

$$F(r,a,b,\beta,\gamma,\delta) = \begin{cases} \frac{l_2}{l_1}\,\mu\,(r,a,b) + \frac{l_3}{l_1}\,\sigma_1(r,\beta) + \frac{l_4}{l_1}\,\sigma_1(r,\gamma) - \frac{l_5}{l_1}\,\eta(r,\delta) & 0 \le r \le r(\gamma) \\ \frac{l_2}{l_1}\,\mu\,(r,a,b) + \frac{l_3}{l_1}\,\sigma_1(r,\beta) + \frac{l_4}{l_1}\,\sigma_2(r,\gamma) - \frac{l_5}{l_1}\,\eta(r,\delta) & r(\gamma) \le r \le r(\beta) \\ \frac{l_2}{l_1}\,\mu\,(r,a,b) + \frac{l_3}{l_1}\,\sigma_2(r,\beta) + \frac{l_4}{l_1}\,\sigma_2(r,\gamma) - \frac{l_5}{l_1}\,\eta(r,\delta) & r(\beta) \le r < 1 \end{cases}$$

and

$$F(r,a,b,0,0,\delta) = \frac{l_2}{l_1}\,\mu\,(r,a,b) + \frac{l_3+l_4}{l_1}\,\sigma_1(r,0) - \frac{l_5}{l_1}\,\eta\,(r,\delta).$$

Then f is univalent and starlike for $|z| < r_0 < 1$ where r_0 is the smallest positive root of the equation $F(r,a,b,\beta,\gamma,\delta) = 0$.
This result is sharp.

THEOREM 4: Let $f(z)^{l_1} = \dfrac{s(z)^{l_2}\,s_1(z)^{l_3}\,u(z)^{l_4}}{v(z)^{l_5}}$ where $s \in S(m,M)$,

$s_1 \in S^*(\alpha)$, $0 \le \alpha < 1$, $u \in P(\beta)$, $0 \le \beta \le 1$, $v \in P(\gamma)$, $0 \le \gamma \le 1$ and l_1, l_2, l_3, l_4, l_5 are all positive real numbers. Let $F(r,a,b,\alpha,\beta,\gamma)$ be defined by

$$F(r,a,b,\alpha,\beta,\gamma) = \begin{cases} \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\nu(r,\alpha) + \frac{l_4}{l_1}\sigma_1(r,\beta) - \frac{l_5}{l_1}\eta(r,\gamma) & 0 \leq r \leq r(\beta) \\ \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\nu(r,\alpha) + \frac{l_4}{l_1}\sigma_2(r,\beta) - \frac{l_5}{l_1}\eta(r,\gamma) & r(\beta) \leq r < 1 \end{cases}$$

and

$$F(r,a,b,\alpha,0,\gamma) = \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\nu(r,\alpha) + \frac{l_4}{l_1}\sigma_1(r,0) - \frac{l_5}{l_1}\eta(r,\gamma) \qquad 0 \leq r < 1.$$

Then f is univalent and starlike for $|z| < r_0 < 1$ where r_0 is the smallest positive root of the equation $F(r,a,b,\alpha,\beta,\gamma) = 0$.

This result is sharp.

THEOREM 5: Let $f(z)^{l_1} = \dfrac{s(z)^{l_2}\, u(z)^{l_3}}{v(z)^{l_4}\, w(z)^{l_5}}$ where $s \in S(m,M)$,

$u \in P(\beta)$, $0 \leq \beta \leq 1$, $v \in P(\gamma)$, $0 \leq \gamma \leq 1$, $w \in P(\delta)$, $0 \leq \delta \leq 1$ and l_1, l_2, l_3, l_4, l_5 are all positive real numbers. Let

$F(r,a,b,\beta,\gamma,\delta)$ be defined by

$$F(r,a,b,\beta,\gamma,\delta) = \begin{cases} \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\sigma_1(r,\beta) - \frac{l_4}{l_1}\eta(r,\gamma) - \frac{l_5}{l_1}\eta(r,\delta) & 0 \leq r \leq r(\beta) \\ \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\sigma_2(r,\beta) - \frac{l_4}{l_1}\eta(r,\gamma) - \frac{l_5}{l_1}\eta(r,\delta) & r(\beta) \leq r < 1 \end{cases}$$

and

$$F(r,a,b,0,\gamma,\delta) = \frac{l_2}{l_1}\mu(r,a,b) + \frac{l_3}{l_1}\sigma_1(r,0) - \frac{l_4}{l_1}\eta(r,\gamma) - \frac{l_5}{l_1}\eta(r,\delta), \qquad 0 \leq r < 1.$$

Then f is univalent and starlike for $|z| < r_0 < 1$ where r_0 is the smallest positive root of the equation $F(r,a,b,\beta,\gamma,\delta) = 0$.

This result is sharp.

THEOREM 6: Let $f(z)^{l_1} = \dfrac{s(z)^{l_2}\, s_1(z)^{l_3}}{u(z)^{l_4}\, v(z)^{l_5}}$ where $s \in S(m,M)$,

$s_1 \in S^*(\alpha)$, $0 \leq \alpha < 1$, $u \in P(\beta)$, $0 \leq \beta \leq 1$, $v \in P(\gamma)$, $0 \leq \gamma \leq 1$ and l_1, l_2, l_3, l_4, l_5 are all positive real numbers. Let $F(r,a,b,\alpha,\beta,\gamma)$ be defined by

$$F(r,a,b,\alpha,\beta,\gamma) = \frac{l_2}{l_1}\,\mu\,(r,a,b) + \frac{l_3}{l_1}\,\nu\,(r,\alpha) - \frac{l_4}{l_1}\,\eta\,(r,\beta) - \frac{l_5}{l_1}\,\eta(r,\gamma).$$

Then f is univalent and starlike for $|z| < r_0 < 1$ where r_0 is the smallest positive root of the equation $F(r,a,b,\alpha,\beta,\gamma) = 0$.
This result is sharp.

THEOREM 7: Let $f(z)^{l_1} = \dfrac{s(z)^{l_2}}{s_1(z)^{l_3}\, u(z)^{l_4}\, v(z)^{l_5}}$ where $s \in S(m,M)$,

$s_1 \in P(\alpha)$, $0 \leq \alpha \leq 1$, $\mu \in P(\beta)$, $0 \leq \beta \leq 1$, $v \in P(\gamma)$, $0 \leq \gamma \leq 1$, l_1, l_2, l_3, l_4, l_5 are all positive real numbers. Let $F(r,a,b,\alpha,\beta,\gamma)$ be defined by

$$F(r,a,b,\alpha,\beta,\gamma) = \frac{l_2}{l_1}\,\mu\,(r,a,b) - \frac{l_3}{l_1}\,\eta\,(r,\alpha) - \frac{l_4}{l_1}\,\eta(r,\beta) - \frac{l_5}{l_1}\,\eta(r,\gamma).$$

Then f is univalent and starlike for $|z| < r_0 < 1$ where r_0 is the smallest positive root of the equation $F(r,a,b,\alpha,\beta,\gamma) = 0$.
This result is sharp.

The proofs of theorems 2 to 7 are similar to that of Theorem 1 and will be omitted here.

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