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On The Restricted Ideal Sheaves

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On The Restricted Ideal Sheaves

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SUMMARY

It is shown,in particular, that the fundamental groups of the restricted ideal sheaves over the complex analytic manifold X, which are normal subgroups of the fundamental group F of X, satisfy the descending (minimal) chain condition.

In this paper we shall expand on the relationship of the homology group of the complex analytic manifold X to the Cohomology group $H^0(X, \mathcal{A})$ of the restricted structure sheaf $\mathcal{A}$ by analyzing more closely the connection of the algebraic structure of the ring A(X) of holomorphic functions on X to that of the restricted sheaf $\mathcal{A}$. [1]. These considerations will ultimately lead to Riemann-Roch Theorem.

The paper quoted in [1] will be refered to as HG.

1. Restricted Sheaves.

Let X be a connected complex analytic manifold with fundamental group F $\neq$ 1 (1 is the identity element). The totality of holomorphic functions on X is denoted as usual by A (X). It is a ring (or C-algebra). As in HG, we make A(X) into a covering topological space $\mathcal{A}$ of X as follows:

Let f $\in$ A(X), and x $\in$ X a point. f can be expanded into a power series f_x convergent at x. The totality of such power series at x as f runs through A(X) is denoted by A_x which is again a ring (C-algebra) isomorphic to A(X). The disjoint union

$$\mathcal{A} = \bigvee_{x \in X} A_x$$

is a set over X with a natural projection

$$\pi : \mathcal{A} \to X$$

mapping each f_x onto the point of expansion x.

We introduce in $\mathcal{A}$ a natural topology as in HG.

Sections in $\mathcal{A}$ are introduced in the usual way. Namely, if $U \subset X$ is an open set, then the continuous mapping

$$s: U \to \mathcal{A}$$

with $\pi \circ s = 1_U$ is called a section of $\mathcal{A}$ over U.

The totality of sections over U is denoted by $\Gamma(U, \mathcal{A})$.

If $s \in \Gamma(U, \mathcal{A})$ then π: s (U) $\to$ U is topological.

Moreover every $s \in \Gamma(U, \mathcal{A})$ can be extended holomorphically to a global section in $\mathcal{A}$ over X.

Definitions:

$\mathcal{A}$, with the natural topology thus introduced is called the restricted sheaf over X. The elements of $\mathcal{A}$ are the convergent power series called germs of holomorphic functions $f \in A(X)$. The points over x form the ring (C-algebras) $A_x = \pi^{-1}(x)$ of germs at x called a stalk of the restricted sheaf $\mathcal{A}$. Any two stalks are of course isomorphic. Moreover, $\mathcal{A}$ is a complete regular covering space of X.

$\Gamma(X, \mathcal{A})$ is an abelian group, and we have [2]

Theorem 1.1. $A(X) \cong \Gamma(X, \mathcal{A})$.

Proof. Let $\gamma: A(X) \to \Gamma(X, \mathcal{A})$ defined by $\gamma f = s$ over X.

1. γ is injective, i.e., Ker $\gamma = 0$. Let $f \in A(X)$. If $\gamma(f) = 0$ then for every $x \in X$ we have $\gamma f(x) = 0$. Namely $s(x) = 0$. Therefore $f(x)=0$. Hence there exists a neighborhood $U(x) \subset X$ such that $s|U = 0$. This means that $f|U = 0$. Therefore $f = 0$ over X.

2. γ is surjective. If $s \in \Gamma(X, \mathcal{A})$ then for every $x \in X$, there exist a neighborhood $U(x) \subset X$ and $f \in A(X)$ with $f|U \ni \gamma(f|U)(x) = (\gamma f|U)(x) = s(x)$. Therefore there is a neighborhood $V(x) \subset U \ni$

$\gamma f|V = s|V$. Now, let $(U_i)_{i\varepsilon I}$ be an open covering of X such that there is $f_i \in A(X)$ with $f_i|U_i \ni \gamma f_i|U_i = s|U_i$. This implies the existence of $f \in A(X) \ni f|U_i = f_i|U_i$ and for which

$$\gamma f|U_i = \gamma f_i|U_i = s|U_i.$$

Hence $\gamma f = s$ over X. The proof is thus complete.

2. Restricted Ideal Sheaves.

Definition 2.1. A restricted analytic sheaf over X is a sheaf S of A-modules over X. [3].

1. A itself is a restricted analytic sheaf.

2. Let S be a restricted analytic sheaf, $S^* \subset S$ a subsheaf. If for every $x \in X$, $S_x^* \subset S_x$ is a submodule, then S^* is also a r-analytic sheaf.

If $(s_1, s_2) \in \Gamma(X, S^* \oplus S^*) \subset \Gamma(X, S \oplus S)$, then $s_1 + s_2 \in \Gamma(X, S)$ and so $s_1 + s_2 \in \Gamma(X, S^*)$. Thus addition is continuous so is Multiplication by scalars. Note that if $S^* \subset S$ is a restricted analytic subsheaf then $\Gamma(X, S^*) \subset \Gamma(X, S)$ is a $\Gamma(X, A)$ submodule.

3. Now, if $I \subset A$ is a r-analytic subsheaf, then $I_x \subset A_x$ is an ideal. For this reason I is called a restricted ideal sheaf, in short r-ideal sheaf. Here the I_x's are isomorphic. In fact,

Theorem 2.1. There is a one to one correspondence between the ideals of A(X) and the r-ideal sheaves.

Proof. Let $I \subset A(X)$ be an ideal, then $I_x \subset A_x$ for every x. Therefore $I = \bigvee_{x \varepsilon X} I_x$ is the ideal sheaf determined by I. Conversely if I is given, then each $I_x \subset I$ defines the ideal $I \subset A(X)$.

Definition 2.2. A r-ideal sheaf I is called proper if I is different from the zero sheaf and A. We conclude that a proper r-ideal sheaf cannot contain the unit section. For, the latter generates the whole A.

Definition 2.3. A r-ideal sheaf J is maximal if it is proper and if $J \subset I \subset A$, then $I = J$. I is any proper r-ideal sheaf.

Theorem 2.2. Every r-ideal sheaf I is contained in a maximal r-ideal sheaf J.

Proof. Order partially by set inclusion the collection P of all proper r-ideal sheaves of A containing $I \subset A$. The natural union of any chain in P is a proper r-ideal sheaf because no r-ideal sheaf of P contains the unit section. In view of Hausdorff Maximality theorem P contains a maximal chain Q. The union of Q is a proper r-ideal sheaf *J*. The maximality of Q implies that *J* is maximal.

3. **r-ideal sheaves as a covering space of X.**

Again from HG we infer that the r-ideal sheaves *I* are complete regular covering spaces of X, and that the group T of cover transformations of *I* is isomorphic to the abelian group $\Gamma(X, I)$ whose elements are uniquely determined by the points (germs) on $I_{x0} \subset A_{x0}$ where $x_0 \in X$ is a fixed point. We have seen, theıe, that the fundamental group of *I* projects onto and is isomorphic to a normal subgroup D of F or its conjugate subgroups in F.

Accordingly,

$$\Gamma(X, I) \cong F/D.$$

Conversely, if D is a normal subgroup of F such that F/D is commutative then D determines a r-ideal sheaf *I* whose fundamental group is isomorphic to D. Therefore, we can state

Theorem 3.1. There is a one to one correspondence between the r-ideal sheaves *I* (or the ideals I of A(X)) and the normal subgroups D of F for which F/D is commutative. Moreover every pair $I \subset I'$ maps onto the pair $D \supset D'$.

If we define a proper normal subgroup as being different from F and [F, F], then a proper minimal normal subgroup D_m of F, such that F/D_m is commutative, is that one for which if for any proper normal subgroup $D \subset F$ such that F/D is commutative $[F, F] \subset D \subset D_m$ then $D = D_m$. By theorem 3.1 it is clear that D_m is isomorphic to the fundamental group of a maximal r-ideal sheaf *J*.

Theorem 3.2. If D is a proper normal subgroup of F such that F/D is commutative then there exists a D_m with same qualification such that $D \supset D_m$. Namely, these D's satisfy the descending (minimum) chain condition.

Pıoof. By hypothesis D is isomorphic to the fundamental group of the r-ideal sheaf *I*. In view of theorem 2.2 $I \subset J$. But by theorem 3.1, $I \subset J$ maps onto $D \supset D_m$.

We finally note that A as a covering space of X is itself a connected complex analytic manifold with fundamental group isomorphic to [F, F]. Yet, in view of the property of [F, F] being the smallest normal subgroup for which F / [F, F] is commutative, the covering space of A would consist of a single section homeomorphic to A. Namely, it is A itself.

4. **The Riemann-Roch Theorem.** With regard to connected Riemann Surfaces X with structure sheaf A, the author's Theory of restricted analytic sheaves when viewed as topological covering spaces of X yields at once the crucial results on the dimensions of the vector spaces of holomorphic functions on these Riemann surfaces, and the Riemann-Roch Theorem thereof.

In particular, theorem 6.4 in HG reads:

Theorem 4.1. On a Riemann Surface with structure sheaf A and genus g there are exactly 2g linearly independent global sections (holomorphic functions). Namely, $\dim_C A(X) = g$.

Proof. The abelianized fundamental group F / [F,F] has exactly 2g linearly independent generators. Now, the theorem follows from the relationship $H^0(X, A) \simeq \Gamma(X, A) \simeq F/[F, F]$. [1].

ÖZET

Özel olarak gösteriliyor ki, F nin normal altgrupları olan tahditli ideal demetlerin esas grupları azalan (minimum) zincir şartını sağlamaktadırlar.

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