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APPROXIMATION OF FUNCTION BY A GENERALIZED SZASZ OPERATORS

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ABSTRACT

In the present paper a modification of Szasz Mirakyan operators is introduced and studied its approximation properties for $f \in [0, \lambda]$, where λ is an arbitrary positive number.

INTRODUCTION

Jain [2] devoted his study to the approximation of a function $f(x) \in C\ [0, \lambda]$ by the linear positive operators defined in (2 .11). Further we modify the operator (2 .11) as in (2 .12).

2. THE OPERATOR AND ITS CONVERGENCE

The operator and its convergence are based on the following two lemmas:

Lemma (2.1). For $0 < \alpha < \infty$, $|\beta| < 1$, let

(2 .1) $w_\beta(k,\alpha) = \alpha(\alpha + k\beta)^{k-1}\ e^{-(\alpha+k-\beta)}\ /\ k, \quad k = 0,1,2,\ldots$

then

(2 .2) $\sum\limits_{k=o}^{\infty} w_\beta(k, \alpha) = 1.$

It may be mentioned that (2 .1) is a Poisson type distribution which has been considered by Consul and Jain [1].

The proof of the lemma may be based upon results given by Jensen [3]. If we start with Lagrange's formula

$$(2.3)\quad \Phi(z) = \Phi(0) + \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{d^{k-1}}{dz^{k-1}} \left((f(z))^k \Phi'(z) \right)\right]_{z=0} \left(\frac{z}{f(z)}\right)^k$$

and proceed by setting

$$\Phi(z) = e^{\alpha z} \text{ and } f(z) = e^{\beta z}$$

we shall get

$$(2.4)\quad e^{\alpha z} = \sum_{k=0}^{\infty} \alpha(\alpha+k\beta)^{k-1} \frac{u^k}{k}, \; u = ze^{-\beta z},$$

where z and u are sufficiently small such that $|\beta u| < e^{-1}$ and $|\beta z| < 1$.

By taking $z = 1$, the lemma (2.1) is obvious.

Lemma (2.2): Let

$$(2.5)\quad S(r,\alpha,\beta) = \sum_{k=0}^{\infty} (\alpha+\beta k)^{k+r-1} e^{-(\alpha+\beta k)/k} \quad r = 0,1,2,...$$

and

$$(2.6)\quad \alpha\, S(0, \alpha, \beta) = 1,$$

then

$$(2.7)\quad S(r,\alpha,\beta) = \sum_{k=0}^{\infty} \beta^k (\alpha+k\beta)\, S(r-1, \alpha+k\beta, \beta).$$

Proof: It can easily be seen that the functions $S(r,\alpha,\beta)$ satisfy the reduction formula

$$(2.8)\quad S(r,\alpha,\beta) = \alpha\, S(r-1,\alpha,\beta) + \beta S(r, \alpha+\beta, \beta).$$

By a repeated use of (2.8), the proof of the lemma is straightforward

From (2.6) and (2.7) when $\beta < 1$ we get

$$(2.9)\quad S(1,\alpha,\beta) = \sum_{k=0}^{\infty} \beta^k(\alpha+k\beta)\, S(0,\alpha+k\beta, \beta)$$

$$= \sum_{k=0}^{\infty} \beta^k = \frac{1}{1-\beta}$$

and

$$(2.10)\quad S(2,\alpha,\beta) = \sum_{k=o}^{\infty} \beta^k (\alpha+k\beta)\, S(1,\alpha+k\beta, \beta)$$

$$= \frac{\alpha}{(1-\beta)^2} + \frac{\beta^3}{(1-\beta)^3}$$

Jain [2] defined the operators as

$$(2.11)\quad P_n^{[\beta]}(f, x) = \sum_{k=o}^{\infty} w_\beta(k, nx)\, f(k/n),$$

where $1 > \beta \geqq 0$ and $w_\beta(k, nx)$ has been defined in (2.1).

The parameter β may depend on the natural number n. For $\beta = 0$ (2.11) reduces to

$$P_n^{[o]}(f,x) = \sum_{k=o}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

as defined by Mirakyan [6] and Szasz [9]. Now, we modify the operator (2.11) as follows

$$(2.12)\quad K_n^{[\beta]}(f,x) = \sum_{k=o}^{\infty} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt\right) w_\beta(k, nx),$$

for $\beta = 0$ (2.12) reduces to operators

$$K_n(f,x) = \sum_{k=o}^{\infty} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} e^{-nx} \frac{(nx)^k}{k!} f(t)\, dt\right),$$

defined by Leviatan [5].

The convergence property of the operators $K_n(f,x)$ for $f \in C[0,b]$ is proved in the following teheorem.

Theorem (2.1). If $f \in C[0,b]$ and $\beta \to 0$ as $n \to \infty$ then the sequence $K_n^{[\beta]}(f,x)$ converges uniformly to f(x) in [a, b], where $0 \leqq a < b < \infty$.

Proof. Since $K_n^{[\beta]}(f,x)$ is a positive linear operator for $1 > \beta \geqq 0$, it is sufficient, by Korovkin's result, to verify the uniform convergence for test functions $f(t) = 1$, t and t^2.

It is clear from (2 .2) that

(2 .14) $K_n^{[\beta]}(1,x) = 1.$

Going on to $f(t) = t$ and using (2 .9) we have

$$(2.15)\quad K_n^{[\beta]}(t,x) = \sum_{k=o}^{\infty} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} t\, dt \right) w_\beta(k, nx)$$

$$= \sum_{k=o}^{\infty} \frac{k}{n} w^\beta(k, nx) + \frac{1}{2n} \sum_{k=o}^{\infty} w_\beta(k, nx)$$

$$= xS(1, nx + \beta, \beta) + \frac{1}{2n}$$

$$= \frac{x}{1+\beta} + o\left(\frac{1}{n}\right)$$

Proceeding to the function $f(t) = t^2$, it can easily be shown that

$$(2.16)\quad K_n^{[\beta]}(t^2,x) = \sum_{k=o}^{\infty} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} t^2 dt \right) w_\beta(k, nx)$$

$$= \sum_{k=o}^{\infty} \frac{k^2}{n^2} w_\beta(k, nx) + \frac{1}{n} \sum_{k=o}^{\infty} \frac{k}{n} w_\beta(k,nx)$$

$$+ \frac{1}{3n^2} \sum_{k=o}^{\infty} w_\beta(k, nx)$$

$$= \frac{x}{n} (S(2,nx+2\beta, \beta) + S(1, nx+\beta, \beta)]$$

$$+ \frac{1}{n}\left[\frac{x}{1-\beta}\right] + \frac{1}{3n^2}$$

$$= \frac{x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3} + \frac{x}{n(1-\beta)} + \frac{1}{3n^2}$$

Thus combining the results of (2 .14), (2 .15) and (2 .16), we have

$$\lim_{n\to\infty} K_n^{[\beta]}(t^r,x) = x^r,\ r = 0,1,2, \text{ as } \beta \to 0.$$

3. ORDER OF APPROXIMATION

Theorem (3.1). If $f \in C\,[0, \lambda]$ and $1 > \beta'/n \geqq \beta \geqq 0$ then

$$|f(x) - K_n^{(\beta)}(f,x)| \leqq [1+\lambda^{1/2}(1+\lambda\beta\beta')^{1/2}],\ w\left(\frac{1}{\sqrt{n}}\right)$$

where $w(\delta) = \sup |f(x'') - f(x')|$, $x', x'' \in [0,\lambda]$, δ being a positive number such that $|x'' - x'| < \delta$. For the proof of Theorem (3.1) we need the following lemma:

Lemma (3.1). For $x \in [0,\lambda]$, where λ is arbitrary positive number and $1 > \frac{\beta'}{n} \geqq \beta \geqq 0$, we have

$$\sum_{k=0}^{\infty} n\, w_\beta(k, nx) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (x-t)^2\, dt \leqq \lambda[1+\lambda\beta\beta'] + o\left(\frac{1}{n}\right)$$

Proof. By linearity of operator by using (2.14), (2.15) and (2.16) we have

$$(3.1)\quad \sum_{k=0}^{\infty} n\, w_\beta(k, nx) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (x-t)^2\, dt$$

$$= x^2 K_n^{[\beta]}(1,x) - 2x\, K_n^{[\beta]}(t,x) + K_n^{[\beta]}(t^2,x)$$

$$= x^2 - \frac{2x^2}{1-\beta} - \frac{x}{n} + \frac{x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3}$$

$$+ \frac{x}{n(1-\beta)} + \frac{1}{3n^2}$$

$$= \frac{x^2\beta^2}{1-\beta^2} + \frac{x}{n(1-\beta)^3} + \frac{x}{n(1-\beta)} - \frac{x}{n} + \frac{1}{3n^2}$$

$$= \frac{x^2\beta^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3} + \frac{x^3}{n(1-\beta)} + \frac{1}{3n^2}$$

$$\leq \frac{x^2\beta^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3} + \frac{1}{3n^2}$$

$$\leq \lambda \left[\frac{\lambda\beta\beta'}{(1-\beta)^2} + \frac{1}{(1-\beta)^3}\right] / n + o\left(\frac{1}{n}\right)$$

$$\leq \lambda \ [1 + \lambda\beta\beta'] / n + o\left(\frac{1}{n}\right).$$

Proof of Theorem (3.1): By using the properties of modulus of continuity

(3.2) $|f(x'') - f(x')| \leq w(|x'' - x'|)$,

(3.3) $w(\gamma\delta) \leq (\gamma + 1)\, w(\delta),\ \gamma > 0$

and noting the fact that

$$\sum_{k=o}^{\infty} w_\beta(k, nx) = 1 \text{ and } w_\beta(k, nx) \geq 0,\ \forall n,k.$$

It can easily be seen, by the application of Cauchy's inequality, that

(3.4) $$|f(x) - K_n^{[\beta]}(f,x)| \leq 1 + \frac{1}{\delta} \sum_{k=o}^{\infty} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} (x-t)dt\, w_\beta(k,nx)\, w(\delta)\right.$$

$$\leq \left\{1 + \frac{1}{\delta}\left[\sum_{k=o}^{\infty} \int_{\frac{k}{n}}^{\frac{k+1}{n}} w_\beta(k,\ nx)\ (x-t)^2 dt\right]^{\frac{1}{2}}\right\} w(\delta)$$

Using Lemma (3.1) in (3.4) and choosing $\delta = \dfrac{1}{\sqrt{n}}$, we prove

(3.5) $$|f(x) - K_n^{[\beta]}(f,x)| \leq [1 + \lambda^{1/2}(1+\lambda\beta\beta')^{1/2}]\, w\left(\frac{1}{\sqrt{n}}\right) + o\left(\frac{1}{n}\right).$$

For $\beta = 0$, the expression (3.5) reduces to an inequality for Szasz–Mirakjan operator obtained earlier by Muller [7].

Theorem (3.2). If $f \in C'[0,\lambda]$ such that its first derivative is bounded on $[0, \lambda]$ and $1 > \beta'/n \geq \beta \geq 0$, then the following inequality holds

$$[f(x) - K_n^{[\beta]}(f, x)] \leq \lambda^{1/2}\ (1 + \lambda\,\beta\beta')^{1/2}\ (1 + \lambda\,\beta\beta')^{1/2}]$$

$$.2\ w_1\left(\frac{1}{\sqrt{n}}\right) / \sqrt{n}$$

Proof. For definiteness, we prove the theorem for $f'(x) \geqq 0$ but it also applies to $f'(x) < 0$. By the mean value theorem of differential calculus, it is known that

$$f(x) - f(t) = (x - t) f'(\xi),$$

where $\xi = \xi_{n,k}(x)$ is an interior point of the interval determined by x and t.

Now

$$f(x) - f(t) \leqq (x-t) \, |f'(\xi) - f'(x)| + (x-t) f'(x)$$

Multiplying both sides of the inequality by n $w_\beta(k, nx)$, integrating from $\frac{k}{n}$ to $\frac{k+1}{n}$ and summing over k we get

$$(3.6) \quad |f(x) - K_n^{[\beta]}(f,x)| \leqq \sum_{k=0}^{\infty} n \int_{\frac{k}{n}}^{\frac{k+1}{n}} |x-t| \, w_\beta(k, nx) \, dt \cdot |f'(\xi) - f'(x)| .$$

But by (3.1) and (3.2)

$$|f'(\xi) - f'(x)| \leqq w_1(|\xi - x|) \leqq \left(1 + \frac{1}{\delta} |\xi - x|\right) w_1(\delta)$$

$$\leqq \left(1 + \frac{1}{\delta} |t-x|\right) w_1(\delta):$$

where δ is a positive number not depending on k.

A use of this in (3.6) gives

$$|f(x) - K_n^{[\beta]}(f,x)| \leqq \left\{ \sum_{k=0}^{\infty} n \int_{\frac{k}{n}}^{\frac{k+1}{n}} |x-t| \, w_\beta(k,nx) \, dt + \frac{1}{\delta} \sum_{k=0}^{\infty} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (x-t)^2 w_\beta(k,nx) dt \right\} w_1(\delta).$$

Hence by applications of Cauchy's inequality and using Lemma (3.1)

$$(3.7)\ |f(x)-K_n^{[\beta]}(f,x)| \leq \frac{\lambda^{1/2}(1+\lambda\beta\beta')^{1/2}}{n^{1/2}}\left[1+\frac{\lambda^{1/2}(1+\lambda\beta\beta')}{\delta n^{1/2}}\right]^{\frac{1}{2}} w_1(\delta) + o\left(\frac{1}{n}\right).$$

Choosing $\delta = \frac{1}{\sqrt{n}}$, Theorem (3.2) is proved.

We may put $\beta = 0$, $\delta = \frac{1}{\sqrt{n}}$ in (3.7) to get the expression for Szasz–Mirakyan operator, the substitutions reduce (3.7) to

$$|f(x)-K_n^{[o]}(f,x)| \leq \frac{1}{\sqrt{n}}(\lambda+\sqrt{\lambda})\ w_1\left(\frac{1}{\sqrt{n}}\right),\ x \in [0,\lambda]$$

in agreement with Stancu [8].

REFERENCES

1– P.C. CONSUL and G.C. JAIN., On a generalized Poisson distribution; Submitted (1970).

2– G.C. JAIN, Approximation of functions by a new class of linear positive operators, Jour. Austral. Math. Soc., 13 (3) 1972, 271–276.

3– J.L.W.V. JENSEN., Sur une identité, A′ Abel et sur d′ autres formules analogues, Acta Math., 26, (1902), 307–318.

4– P.P. KOROVKIN, Linear Operators and Approximation theory (translated from Russian edition of 1959, Delhi 1960).

5– D. LEVIATAN, The rate of approximation in the L_1–norm by some Bernstein–type operators. Unveroffentlicht 1972.

6– G. MIRAKYAN., Approximation des Fonctions continues au Moyen de Polynomes de la Forme, Dokl. Akad. Nauk. SSSR 31 (1941), 201–205.

7– M. MULLER, Die Folge der Gamma operatoren, Dissertation, Stuttgart, 1967.

8– D.D. STANCU, Approximation of functions by a new class of linear Polynomial operators, Revue Roumaine de Mathematique Pures et Appliquees 8 (XIII) (1968), 1173 – 1194.

9– O. SZASZ., Generalization of S. Bernstein's polynomials to the infinite interval, J. of Research of the Nat. Bur. of Standards., 45, (1950), 239–245.