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APPROXIMATION BY MODIFIED BERNSTEIN OPERATORS

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APPROXIMATION BY MODIFIED BERNSTEIN OPERATORS

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ABSTRACT

In this paper, our object is to prove analogous theorems concerning the degree of approximation by operators $K_n^{(\beta)}(f,x)$ corresponding to those obtained by Jain and Pethe [1] for the operators $P_n^{(\beta)}(f,x)$.

INTRODUCTION

Various generalizations of the well known Bernstein operator B_n, defined on C [0,1] by the relation

$$(1.1)\quad B_n(f,x) = \sum_{k=0}^{n} b_{n,k}(x)\, f\left(\frac{k}{n}\right),$$

where

$$(1.2)\quad b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\quad k=0,1,\ldots,n$$

have recently been found. The Bernstein operator itself and many of its generalizations are built by starting from the identity

$$\sum_{k=0}^{n} b_{n,k}(x) = 1.$$

(1.2) is the well known binomial distribution. In a sequence of independent trials, let P_k be the probability of a success after k previous successes. If $P_k = x$, it is known that the probability of exactly k successes in n trials is $b_{n,k}(x)$, $k = 0,1,\ldots,n$. It is obvious that by taking a more general value for P_{nk}, a corresponding generalization of the operators B_n will be obtained. Jain and et al [1] gave such generalization of B_n defined by (1 .1).

2 . Operators: Let $P_k = x + \beta k$, where β is any real number satisfying the condition $0 < P_k < 1$ for all k, k = 0,1,2,..., n. The probability $P_k(n,\beta)$ of exactly k successes in n trials is then given by

$$(2.1)\quad P_k(n,\beta) = \frac{1}{k!} \prod_{i=0}^{k-1} \left(\frac{X}{\beta} + i\right) \sum_{r=0}^{k} (-1)^r \binom{k}{r} (1-x-\beta r)^n,$$

where $\prod_{i=0}^{k-1} \left(\frac{X}{\beta} + i\right)$ is defined as 1, when $k = 0$.

Jain et al [1] defined on C [a,b], $0 \leq a < b < 1$, the sequence of the operators $P_n^{(\beta)}$ by the relation

$$(2.2)\quad P_n^{(\beta)}(f,x) = \sum_{k=0}^{n} P_k(n,\beta) f\left(\frac{k}{n}\right),$$

where β may depend on the number n. Since $0 < P_k < 1$. the limiting conditions on β are

$$\beta < (1-x)/n \text{ if } \beta \geq 0 \text{ and } |\beta| < \frac{x}{n}, \text{ if } \beta < 0.$$

It is easily seen that the operator $P_n^{(\beta)}$ approaches that of Bernstein when $\beta \to 0$, and is actually Bernstein when $\beta = 0$. Indeed when $\beta = 0$, $P_k = x$ and $P_k(n,0) = \binom{n}{k} x^k (1-x)^{n-k}$.

Now, we define a Kantorovitch type polynomial with the help of (2 .2) for Lebesgue integrable functions on [0,1] in L_1–norm as

$$(2.3)\quad K_n^{(\beta)}(f,x) = \sum_{k=0}^{n} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \; P_k(n,\beta),$$

where $P_k(n, \beta)$ is defined as in (2.1).

If we put $\beta = 0$ then (2 .3) reduces to the operator

$$K_n(f,x) = (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} x^k(1-x)^{n-k} f(t)dt.$$

3. Convergence: With regards to the convergence of the sequence of polynomials $K_n^{(\beta)}(f,x)$ fcr $f(x) \in C[0,1]$, we state and prove the folowing:

Theorem (3.1): If $f \in C[0,1]$ and if $\beta \to 0$ as $n \to \infty$, so that $n\beta \to 0$, then the sequence $K_n^{(\beta)}(f,x)$ converges uniformly to $f(x)$ on $[a,b]$, where $0 \leq a < b \leq 1$.

We need Lemma 3.1 to establish Theorem 3.1
Lemma (3.1): The following identities hold.

$$(3.1) \quad \sum_{k=0}^{n} P_k(n,\beta) = 1,$$

$$(3.2) \quad \sum_{k=0}^{n} k\, P_k(n,\beta) = nx\,(1+\beta_1(1)), \text{ and}$$

$$(3.3) \quad \sum_{k=0}^{n} k^2 P_k(n,\beta) = nx\,(1+\beta_1(n)) + n(n-1)\,x\,(x+\beta)\,(1+\beta_2(n)),$$

where

$$\beta_1(n) = \frac{(1+\beta)^n - 1}{n\beta} - 1, \text{ and}$$

(3.4)

$$\beta_2(n) = \frac{1+(1+2\beta)^n - 2(1+\beta)^n}{n(n-1)\beta^2} - 1$$

Proof: From (2.1) ve we have

$$P_k(n,\beta) = \frac{1}{k!} \prod_{i=0}^{k-1} \left(\frac{X}{\beta} + i\right) \text{ X the coefficient of } \frac{\theta^n}{n}$$

$$\text{in } \beta \sum_{n=0}^{\infty} \sum_{r=0}^{k} (-1)^r \binom{k}{r} (1-X-\beta r)^n \frac{\theta^n}{n}$$

$$= \frac{1}{k!} \prod_{i=0}^{k-1} \left(\frac{X}{\beta} + i\right) \text{ X the ceoefficient of } \frac{\theta^n}{n}$$

$$\text{in } \sum_{r=0}^{k} (-1)^r \binom{k}{r} e^{(1-X-\beta r)e}$$

$$= \frac{1}{k!} \prod_{i=0}^{k-1} \left(\frac{X}{\beta} + i\right) X \text{ the coefficient of } \frac{\theta^n}{n} \text{ in}$$

$$e^{(1-X)\theta} \; (1-e^{-\beta\theta})^k.$$

Hence the probability generating function P(t) of k is given by

$$(3.5)\quad P(t) = \sum_{k=0}^{n} t^k P_k(n,\beta)$$

$$= \text{the coefficient of } \frac{\theta^n}{n} \text{ in } e^{(1-X)\theta} \; [1-t\,(1-e^{-\beta\theta})]^{-X/\beta}.$$

Therefore, from (3.5) we have

$$(3.6)\quad \sum_{k=0}^{n} P_k(n,\beta) = P(1) = \text{the coefficient of } \frac{\theta^n}{n} \text{ in } e^{\theta}$$

$$= 1.$$

To prove (3.2), we differentiate (3.5) w.r. to t. On simplifying we get

$$(3.7)\quad \frac{dP(t)}{dt} = \text{the coefficient of } \frac{\theta^n}{n} \text{ in}$$

$$\frac{X}{\beta} \, e^{(1-X)\theta} \; (1-e^{-\beta\theta}) \; [1-t\,(1-e^{-\beta\theta})]^{-\frac{X}{\beta}-1}.$$

Therefore,

$$\sum_{k=0}^{n} k\, P_k(n,\beta) = \left[\frac{dP(t)}{dt}\right]_{t=1} = \text{the coefficient of } \frac{\theta^n}{n}$$

$$\text{in } \frac{X}{\beta} \, e^{\theta} \, (e^{\theta\beta}-1)$$

$$= \frac{X}{\beta} \, [(1+\beta)^n - 1]$$

$$= nX\,(1+\beta_1(n)),$$

where $\beta_1(n)$ is given by (3.4).

Further, differentiating (3 .7) w.r. to t and simplifying, we get

$$\frac{d^2P(t)}{dt^2} = \text{the coefficient of } \frac{\theta^n}{n} \text{ in}$$

$$\frac{X}{\beta}\left(\frac{X}{\beta}+1\right)e^{(1-X)\theta}\,(1-e^{-\beta\theta})^2\,[1-t(1-e^{-\beta\theta})]^{-(X/\beta)-2}$$

Hence

$$(3.8)\quad \sum_{k=0}^{n} k(k-1)P_k(n,\beta) = \left[\frac{d^2P(t)}{dt^2}\right]_{t=1} = \frac{X}{\beta}\left(\frac{X}{\beta}+1\right)[(1+2\beta)^n - 2(1+\beta)^n+1].$$

$$= n(n-1)\,X\,(X+\beta)\left[1+\frac{(1+2\beta)^n-2(1+\beta)^n+1}{n(n-1)\beta^2}-1\right]$$

$$= n(n-1)\,X\,(X+\beta)\,[1+\beta_2(n)]$$

Adding (3 .2) and (3 .8), we get (3 .3).

Proof of Theorem (3 .1): It is easy to see that $K_n^{(\beta)}$ is a linear positive operator. Therefore it is sufficient, by Korovkin's theorem [3], to verify the uniform convergence of $K_n^{(\beta)}(f,x)$ for the test functions $f(t) = 1$, t and t^2.

It is clear from the definition (2 .3) of $K_n^{(\beta)}$ and (3 .6) that

$$(3.9)\quad K_n^{(\beta)}(1,x) = \sum_{k=0}^{n}(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\; P_k(n,\beta)$$

$$= \sum_{k=0}^{n} P_k(n,\beta)$$

$$= 1$$

Observe that under the conditions of the theorem

$$\lim_{n\to\infty}\beta_1(n) = 0 = \lim_{n\to\infty}\beta_2(n).$$

Hence, it is easy to see from (3 .2) and (3 .3) that

$$(3.10)\quad K_n^{(\beta)}(t,x) = \sum_{k=0}^{n}(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t\,dt\; P_k(n,\beta)$$

$$= \frac{1}{(n+1)} \sum_{k=o}^{n} k\, P_k(n,\beta) + \frac{1}{2(n+1)} \sum_{k=o}^{n} P_k(n,\beta)$$

$$= \frac{1}{n+1} \left[nx\, (1+ \beta_1 (n) \right] + \frac{1}{2(n+1)}$$

$$(3.11)\quad K_n^{(\beta)}(t^2,x) = \sum_{k=0}^{n} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t^2 dt\; P_k(n,\beta)$$

$$= \frac{1}{(n+1)^2} \sum_{k=o}^{n} K^2 P_k(n,\beta) + \sum_{k=o}^{n} K\, P_k(n,\beta)$$

$$+ \frac{1}{3} \sum_{k=o}^{n} P_k(n,\beta)$$

$$= \frac{1}{(n+1)^2}\; nx(1+\beta_1(n)) + n(n-1)x(X+\beta)(1+\beta_2(n))$$

$$+ nx\; (1+\beta_1 (n)) + \frac{1}{3}$$

Combining the results of (3 .9), 3(.10) and (3 .11) we have

$$\lim_{n\to\infty} K_n^{(\beta)}\; (t^r, x) \doteq x^r,\; r = 0,1,2.$$

4 . Estimate of order of approximation

Theorem (4 .1). If $f \in C\; [a, b]$, $0 \leq a < b \leq 1$ and $1+2n\beta_1 (n) > (n-1)\; \beta_2 (n)$, then

$$(4.1)\quad |f(x)- K_n^{(\beta)}(f,x) \leq 2w \left(\frac{c}{2dn\sqrt{n}}\right), + 0\left(\frac{1}{n}\right)$$

where

(4 .2) $c = 1 + 2\beta_1 (n) + (n-1)\, (1+\beta_2 (n))\beta$

(4 .3) $d = [1 + 2n\, \beta_1 (n) - (n-1)\, \beta_2 (n)\,]^{1/2}$ and

$w\,(\delta)$ is the modulus of continuity attached to the function f.

For the proof of the Theorem (4.1) we need the following lemma:
Lemma (4.1): For $x \in [a,b]$, $0 \leq a < b < 1$ and $1+2n\beta_1(n) > (n-1)\beta_2(n)$, we have

$$\sum_{k=0}^{n} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} P_k(n,\beta)\,(x-t)^2\,dt \leq \frac{c^2}{4nd^2} + o\left(\frac{1}{n}\right)$$

Proof: By linearity of the operator and using (3.1), (3.2), and (3.3) we have

$$(4.4)\quad (n+1) \sum_{k=0}^{\infty} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} P_k(n,\beta)\,(x-t)^2 dt$$

$$= x^2 K_n^{\beta}(1,x) - 2x\, K_n^{(\beta)}(t,x) + K_n^{\beta}(t,x)$$

$$= x^2 - 2x \left[\frac{nx\,(1+\beta_1(n)}{n+1} + \frac{1}{2(n+1)}\right]$$

$$+ \frac{1}{(n+1)^2}\left[nx\,(1+\beta_1(n)) + n(n-1)\,x\,(X+\beta)\,(1+\beta_2(n)) + nx\,(1+\beta_1(n)) + \frac{1}{3}\right]$$

$$\leq X^2 - 2X \left[\frac{nX(1+\beta_1(n))}{n} + \frac{1}{2n}\right]$$

$$+ \frac{1}{n^2}\left[2nx\,(1+\beta_1(n) + n(n-1)x\,(x+\beta) + \frac{1}{3}\right]$$

$$= \frac{CX - d^2X^2}{n} + o\left(\frac{1}{n}\right)$$

Noting the fact that max $(CX - d^2X^2)$ occurs when $X = \frac{c}{2d^2}$, we get from (4.4) that

$$\sum_{k=0}^{n} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} P_k(n,\beta)\,(x-t)^2 dt \leq \frac{c^2}{4n\,d^2} + o\left(\frac{1}{n}\right)$$

Proof of Theorem (4.1): By using the properties of modulus of continuity, viz.,

$$|f(x') - f(x'')| \leq w(|x'-x''|),$$

and

$$w(\lambda\delta) \leq (\lambda+1)\, w(\delta), \lambda > 0,$$

and by making use of Cauchy's inequality, it can be easily deduced that

$$(4.5) \quad |f(x)-K_n^{(\beta)}(f,x)| \leq \left\{ 1 + \frac{1}{\delta} \left[(n+1) \sum_{k=0}^{n} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} P_k(n,\beta) (x-t)^2 \right]^{\frac{1}{2}} \right\} w(\delta)$$

Using Lemma (4.4) we get

$$(4.6) \quad |f(x)-K_n^{(\beta)}(f,x)| \leq \left\{ 1 + \frac{1}{\delta} \frac{c}{2d\sqrt{n}} \right\} w(\delta) + o\left(\frac{1}{n}\right).$$

Choosing $\delta = \dfrac{c}{2d\sqrt{n}}$ in (4.6), we get (4.1).

Remark. We have noted earlier that when $\beta = 0$, $K_n^{(\beta)}$ reduced to the modified Bernstein operator defined by Kantorovitch [2]. Choosing

$\delta = \dfrac{1}{\sqrt{n}}$ and noting the fact that $c \to 1$ and $d \to 1$ as $\beta \to 0$ (4.1)

reduces, when $\beta \to 0$ to

$$|f(x) - K_n(f,x)| \leq \frac{3}{2} \; w\left(\frac{1}{\sqrt{n}}\right),$$

which agrees with the result proved by Popouiciu [5].

Theorem (4.2). Let $f \in C'[0,1]$. If $1+2n\beta_1(n) > (n-1)\beta_2(n)$ and $\beta \geq 0$, the following inequality holds

$$(4.7) \quad |f(x) - K_n^{\beta}(f,x)| < \frac{c}{d\sqrt{n}} \; w_1\left(\frac{c}{2d\sqrt{n}}\right),$$

where c and d are respectively as defined in (4.32) and (4.3) and $w_1(\delta)$ is the modulus of continuity of f'.

Proof: We prove the theorem for $f'(X) \geq 0$. It can be proved similarly when $f'(x) < 0$. By mean value theorem of differential calculus, we have

$$f(X) - f(t) = (x-t)\, f'(\xi),$$

where $\xi = \xi_{n,k}(x)$ is an interior point of the interval determined by x and t. Since $\beta \geq 0$ it can be easily deduced that

$$(4.8)\quad f(X)-f(t) \leq (x-t)\ |f'(\xi)-f'(x)| + |X(1+\beta_1(n))-t|\, f'(x).$$

Multiplying both sides of (4.7) by $(n+1)\, P_k(n,\beta)$, integrating between the limits $\frac{k}{n+1}$ and $\frac{k+1}{n+1}$, summing over k, using (3.2) and the properties of the modulus of continuity, (4,8) reduces to

$$(4.9)\quad |f(X)-K_n^{(\beta)}(f,x)| \leq \left\{ \left[\sum_{k=0}^{n} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} P_k(n,\beta)\, (x-t)^2 \right]^{\frac{1}{2}} \right.$$

$$\left. \left[1 + \frac{1}{\delta} \left(\sum_{k=0}^{n} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} P_k(n,\beta)\ (X-t)^2 \right]^{\frac{1}{2}} \right\} . w_1(\delta),$$

where δ is a positive number not depending on k. Finally making use of Lemma (4.1) in (4.9) and choosing

$$\delta = \frac{c}{2d\sqrt{n}}$$

we get (4.7).

Remark. As in the remark of Theorem 5.1, in the special case when

$\beta = 0$, we choose $\delta = \frac{1}{\sqrt{n}}$ reducing (4.7) to the inequality:

$$|f(x) - K_n^{(o)}(f,x)| \leq \frac{3}{4} \frac{1}{\sqrt{n}}\ w_1\left(\frac{1}{\sqrt{n}}\right),$$

which is in agreement with the one proved earlier by Lorentz [4] for the Bernstein operator.

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