COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1 : Mathematique

TOME : 34 ANNÉE : 1985

ON A CHARACTERIZATION OF q-LAGUERRE POLYNOMIALS

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2

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications dé la Faculté dés Sciéncés dé l'Univérsité d'Ankara

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(Received June 21, 1985 : Accepted June 26, 1985).

SUMMARY

Carlitz [2] proved that the integral formula of Kogbetlianz [6] together with the relation $L_1^{\alpha}(x) = 1 + \alpha - x$ characterize Laguerre polynomials. The author has proved a q-analogue of the formula of Kogbetliantz [6] and proved a similar characterization for q-Laguerre polynomials on for q-Laguerre polynomials introduced by Jackson and also has given some integral representations of q-Laguerre polynomials.

1. INTRODUCTION

Carlitz [2] showed that the following integral formula characterizes Laguerre polynomials:

$$(1)\ {}_0\int^1 L_m^{(\alpha)}(xt)\ L_n^{(\beta)}((1-x)t)x^{\alpha}(1-x)^{\beta}\,dx$$

$$= \binom{m+n}{m} \frac{\Gamma(\alpha+m+1)\,\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+m+n+2)}\ L_{m+n}^{(\alpha+\beta+1)}(t) \qquad (\alpha,\beta > -1)$$

In the present paper besides giving some q-integral representations of q-Laguerre polynomials, a q-analogue of formula (1.1) has also been obtained which has finally been shown to characterize q-Laguerre polynomials, ${}_qL_n^{(\alpha)}(x)$. The characterization so obtained for q-Laguerre polynomials may be regarded as an extension of the one obtained by Carlitz [2]

2. Notation and definitions: For $0 < q < 1$, let

$$[\alpha] = \frac{1-q^{\alpha}}{1-q},\ (q^{\alpha})_n = (1-q^{\alpha})\ (1-q^{\alpha+1})\ \ldots\ (1-q^{\alpha+n-1}),\ (q^{\alpha})_0 = 1,$$

$$(q^{\alpha})_{\infty} = \prod_{j=0}^{\infty} (1-q^{\alpha+j}),\ \binom{m}{n}_q = \frac{(q)_m}{(q)_n(q)_{m-n}}$$

We then define the following basic hypergeometric functions:

$${}_r\Phi_s{}^{(q)}\left[\begin{matrix}(a_r);x\\(b_s);\lambda\end{matrix}\right] = \sum_{n=o}^{\infty} \frac{(q^{a_1})_n\ldots(q^{a_r})_n}{(q)_n(q^{b_1})_n\ldots(q^{b_s})_n}\; x^n q^{\frac{1}{2}\lambda n(n+1)},$$

when $\lambda = 0$, the function on the left is simply denoted by

$${}_r\Phi_s{}^{(q)}\left[\begin{matrix}(a_r);\ x\\(b_s)\end{matrix}\right]$$

$${}_qL_n{}^{(\alpha)}(x) = \frac{(q^{1+\alpha})_n}{(q)_n}\ {}_1\Phi_1{}^{(q)}\left[\begin{matrix}-n\ ;\ x\\1+\alpha\end{matrix}\right],$$

$${}_qL_n{}^{(\alpha)}(x;1) = \frac{(q^{1+\alpha})_n}{(q)_n}\ {}_1\Phi_1{}^{(q)}\left[\begin{matrix}-n\ ;\ x\\1+\alpha;\ 1\end{matrix}\right]$$

$$\Gamma_q(\alpha) = \frac{(1-q)_{\alpha-1}}{(1-q)^{\alpha-1}} \qquad (\alpha \neq 0, -1, -2, \ldots)$$

Also, in the notation of Hahn [4] if a function can be expressed in the form:

$$f(x) = \sum_{r=o}^{\infty} a_r x^r$$

the function $\sum_{r=o}^{\infty} a_r\,(y-x)_r$ is denoted by $f([y-x])$.

The basic integrals are defined through the relations

$$\int_x^{\infty} f(t)d(t;q) = x(1-q) \sum_{k=1}^{\infty} q^{-k}\ f(x\ q^{-k})$$

$$\int_o^{x} f(t)d(t;q) = x(1-q) \sum_{k=o}^{\infty} q^k f(x\ q^k)$$

$$\int_o^{\infty} f(t)d(t;q) = (1-q) \sum_{-\infty}^{\infty} q^k f(q^k)$$

3. Integral Representations of q-Laguerre Polynomials.

We give two basic integrals involving the q-Laguerre polynomials in terms of generalized basic hypergeometric functions. The results are

(1) $$\frac{1}{(1-q)} \int_0^1 t^{a-1}(1-qt)_{b-1}\, {}_qL_n^{(\alpha)}(xt)d(t;q)$$

$$= \frac{(q)_\infty (q^{a+b})_\infty (q^{1+\alpha})_n}{(q^a)_\infty (q^b)_\infty (q)_n} {}_2\Phi_2 \begin{bmatrix} q^{-n}, & q^a; x \\ q^{1+\alpha}, & q^{a+b} \end{bmatrix}$$

(2) $$\frac{1}{(1-q)} \int_0^1 t^{a-1}\, (1-qt)_{b-1}\, {}_qL_n^{(\alpha)}(x[1-q^b t])d(t;q)$$

$$= \frac{(q)_\infty (q^{a+b})_\infty (q^{1+\alpha})_n}{(q^a)_\infty (q^b)_\infty (q)_n} {}_2\Phi_2 \begin{bmatrix} q^{-n}, & q^b; x \\ q^{1+\alpha}, & q^{a+b} \end{bmatrix}$$

In the left hand side of (3.1), writing the series for ${}_qL_n^{(\alpha)}(x)$, interchanging orders of summation and q-integration valid for $Rl(a)>0$, $Rl(b)>0$ and using the well-known result (Hahn [4]);

(3) $$\frac{1}{(1-q)} \int_0^1 x^{a-1}(1-qx)_{b-1}\, d(x;q) = \frac{(q)_\infty (q^{a+b})_\infty}{(q^a)_\infty (q^b)_\infty}$$

we obtain the required result (3.1) after some simplification.

The proof of (3.2) is similar to that of (3.1).

In particular when $a = 1 + \alpha$, (3.1) yields

(4) $$\frac{1}{(1-q)} \int_0^1 t^{\alpha}(1-qt)_{b-1}\, {}_qL_n^{(\alpha)}(xt)d(t;q)$$

$$= \frac{(q)_\infty (q^{1+\alpha+b+n})_\infty}{(q^{1+\alpha+n})_\infty (q^b)_\infty} {}_qL_n^{(\alpha+b)}(x).$$

Further, if $b = 1$ and α is replaced by $\alpha-1$, (3.4) becomes

(5) $$[\alpha+n] \int_0^1 t^{\alpha-1}\, {}_qL_n^{(\alpha-1)}(xt)d(t;q) = {}_qL_n^{(\alpha)}(x).$$

In fact, (3.4) and (3.5) provide basic integral representation for ${}_qL_n^{(\alpha)}(x)$.

Similarly as particular cases of (3.2), we have by putting $b = 1+\alpha$

(6) $$\frac{1}{(1-q)} \int_0^1 t^{a-1}(1-qt)_\alpha\, {}_qL_n^{(\alpha)}(x\,[1-q^{1+\alpha}t])d\,(t;q)$$

$$\frac{(q)_\infty\,(q^{1+a+\alpha+n})_\infty}{(q^{1+\alpha+n})_\infty\,(q^a)_\infty}\;{}_qL_n^{(a+\alpha)}(x).$$

Further, by taking $a = 1$ and replacing α by $\alpha-1$, (3.6) gives

$$(7)\quad [\alpha+n]\int_0^1 (1-qt)_{\alpha-1}\;{}_qL_n^{(\alpha-1)}(x[1-q^\alpha t])d(t;q) = {}_qL_n^{(\alpha)}(x).$$

Relations (3.6) and (3.7) are also basic integral representations for ${}_qL_n^{(\alpha)}(x)$.

Similar results hold for ${}_qL_n^{(\alpha)}(x,\ 1)$ also.

Now using the following result due to Abdi [1]:

$$(8)\quad \frac{1}{(1-q)}\int_0^{s^{-1}} E_q(qsx)x^{c-1}\;{}_m\Phi_n\;[q^{(a_m)};\; q^{(b_n)};xt]\;d(x;q)$$

$$= s^{-c}\,(q)_{c-1}\;{}_{m+1}\Phi_n\;[q^{(a_m)},\; q^c;\; q^{(b_n)};\; t/s]$$

yet another integral representation for ${}_qL_n^{(\alpha)}(x)$ is given by the following relation:

$$\frac{1}{(1-q)}\int_0^1 E_q(qt)t^c\;{}_1\Phi_2\left[\begin{matrix} q^{-n} & ;xt \\ q^{1+\alpha},\ q^{1+c}; & q\end{matrix}\right] d\,(t;q)$$

$$= \frac{(q)_c\,(q)_n}{(q^{1+\alpha})_n}\;{}_qL_n^{(\alpha)}(x)$$

4. An Integral Formula Involving ${}_qL_n^{(\alpha)}(x)$.

We now give the following q-integral formula involving the q-Laguerre polynomials ${}_qL_n^{(\alpha)}(x)$:

$$(1)\quad {}_0\!\int^1 {}_qL_m^{(\alpha)}(xt)\;{}_qL_n^{(\alpha)}([1-xq^{1+\beta}]\;t\;q^{-m})\;x^\alpha(1-qx)_\beta\;d(x;q)$$

$$= \binom{m+n}{m}_q \frac{\Gamma_q(\alpha+m+1)\,\Gamma_q(\beta+n+1)}{\Gamma_q(\alpha+\beta+m+n+2)}\;{}_qL_{m+n}^{(\alpha+\beta+1)}(t).$$

Proof of (4.1). The left hand side is equal to

$$\frac{(q^{1+\alpha})_m(q^{1+\beta})_n}{(q)_m(q)_n}\sum_{k=0}^{m}\sum_{r=0}^{n}\frac{(q^{-m})_k(q^{-n})_r t^{k+r}q^{-mr}}{(q)_k(q)_r(q^{1+\alpha})_k(q^{1+\beta})_r}$$

$$\times\int_0^1 x^{k+\alpha}(1-xq)_{r+\beta}d(x;q)$$

$$= \frac{(q^{1+\alpha})_m(q^{1+\beta})_n(1-q)}{(q)_m(q)_n} \sum_{k=o}^{m} \sum_{r=o}^{n} \frac{(q^{-m})_k(q^{-n})_r(q)_{k+\alpha}(q)_{r+\beta}t^{k+r}q^{-mr}}{(q)_k(q)_r(q^{1+\alpha})_k(q^{1+\beta})_r(q)_{k+r+\alpha+\beta+1}}$$

$$= \frac{(q)_{m+\alpha}(q)_{n+\beta}(1-q)}{(q)_m(q)_n(q)_{\alpha+\beta+1}} \sum_{k=o}^{m} \sum_{r=o}^{n} \frac{(q^{-m})_k(q^{-n})_r t^{k+r}q^{-mr}}{(q)_k(q)_r(q^{\alpha+\beta+2})_{k+r}}$$

$$= \binom{m+n}{m}_q \frac{(q^{\alpha+\beta+2})_{m+n}}{(q)_{m+n}} \frac{\Gamma_q(\alpha+m+1)\,\Gamma_q(\beta+n+1)}{\Gamma_q(\alpha+\beta+m+n+2)}$$

$$\sum_{j=o}^{m+n} \frac{(q^{-m})_j\; t^j}{(q)_j(q^{\alpha+\beta+2})_j} \sum_{r=o}^{\infty} \frac{(q^{-j})_r(q^{-n})_r q^r}{(q)_r\,(q^{1+m-j})_r}$$

$$= \binom{m+n}{m} \frac{(q^{\alpha+\beta+2})_{m+n}}{(q)_{m+n}} \frac{\Gamma_q(\alpha+m+1)\,\Gamma_q(\beta+n+1)}{\Gamma_q(\alpha+\beta+m+n+2)} \times$$

$$\sum_{j=o}^{m+n} \frac{(q^{-m})_j\; t^j\;(q^{1+m+n-j})_j\; q^{-nj}}{(q)_j(q^{\alpha+\beta+2})_j(q^{1+m-j})_j}$$

$$= \binom{m+n}{m} \frac{\Gamma_q(\alpha+m+1)\,\Gamma_q(\beta+n+1)}{\Gamma_q(\alpha+\beta+m+n+2)}\; {}_qL_{m+n}^{(\alpha+\beta+1)}(t)$$

which is the right hand side of (4.1). This completes the proof of (4.1) which is a q-analogue of a formula due to Feldheim [3]. For $n=0$, formula (4.1) becomes

$$\int_o^1 {}_qL_m^{(\alpha)}\;(xt)\;\; x^\alpha\;\; (1-qx)_\beta\;\; d(x;q)$$

$$= \frac{\Gamma_q(\alpha+m+1)\,\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+m+2)}\; {}_qL_m^{(\alpha+\beta+1)}(t)$$

which is a q-analogue of a formula due to Kogbetliantz [6] In the next article the formula (4.1) will be used to characterize the q-Laguerre polynomials ${}_qL_n^{(\alpha)}(x)$.

5. A Characterization of ${}_qL_n^{(\alpha)}(x)$.

In this article we show that formula (4.1) together with the relation

$${}_qL_1^{(\alpha)}(t) = [1+\alpha] - \frac{x}{(1-q)},$$

characterizes q-Laguerre polynomials ${}_qL_n^{(\alpha)}(x)$. The characterization is given in the form of the following theorem:

THEOREM.

Let $f_{m,q}^{(\alpha)}(t)$ be a q-polynomial in t of degree m and an arbitrary function of the parameter α; also let $f_{m,q}^{(\alpha)}(t)$ satisfy

$$\int_0^1 f_{m,q}^{(\alpha)}(xt)\ f_{n,q}^{(\beta)}\ ([1-q^{1+\alpha}x]\ tq^{-m})\ x^{\alpha}\ (1-qx)_{\beta}\ d(x.q)$$

$$= \binom{m+n}{m}_q \frac{\Gamma_q(\alpha+m+1)\ \Gamma_q(\beta+n+1)}{\Gamma_q(\alpha+\beta+m+n+2)}\ f_{m+n,q}^{(\alpha+\beta+1)}(t)$$

for all m, $n \geq 0$. Then $f^{(\alpha)}_{m,q}(t)$ is given by

$$f_{m,q}^{(\alpha)}(t) = \sum_{r=0}^{m} \frac{C_{0,r}^{m-q}\ C_{1,q}^{r}\ (q^{1+\alpha})_m}{(q)_{m-r}\ (q^{1+\alpha})_r}\ {}_qL_r^{(\alpha)}(t)$$

where $C_{0,q}$ and $C_{1,q}$ are arbitrary constants. In particular, if

$$f^{(\alpha)}{}_{1,q} = {}_qL_1^{(\alpha)}(t) = [1+\alpha] - \frac{x}{(1-q)}$$

then $f^{(\alpha)}{}_{m,q}\ (t) = {}_qL_m^{(\alpha)}(t)$ for all m.

The proof of this theorem is similar to the one given by Carlitz [2] for his characterization of Laguerre polynomials and hence is omitted.

ACKNOWLEDGEMENT

I express my sincere gratidude to my supervisor Prof. R.P. Agarwal, Head of the Department of Mathematics and Astronomy, Lucknow University, Lucknow for his constant help in preparation of this paper. My thanks are also due to Dr. A. Verma for suggesting me the problem and for many valuable discussions we had.

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