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SOME BANACH SEQUENCE SPACES

By

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1. Let E denote a Banach space over the field of real or complex numbers. The spaces $c_0(E)$, $c(E)$, $\ell_p(E)$ and $\ell_\infty(E)$ of sequences in E have been discussed by Boas [1], Day [2], Leonard [9], Maddox [11] and some others. Lorentz [10] introduced the space $\hat{c}$ of almost convergent sequences of real or complex numbers. Subsequently almost convergent sequences have been discussed by Duran [5], King [6], Nanda [14, 15] and some others. The space $\hat{c}$ (E) of almost convergent sequences in E has been discussed by Kurtz [7, 8]. More recently some new sequence spaces, which are denoted by $\hat{\ell}_p$ and $\bar{\ell}_p$, have been introduced in [3] and [4]. The space $\hat{\ell}_p$ of sequences of real or complex numbers emerged from $\hat{c}$ as its absolute analogue, just as ℓ_p emerged from the concept of convergence.

The purpose of this paper is to introduce and study the spaces $\hat{\ell}_p$ (E) and $\bar{\ell}_p(E)$ of sequences in E. This problem naturally comes up for investigation and will fill up a gap in the existing literature. In section 2 we introduce the spaces $\bar{\ell}_p(E)$, $\hat{\ell}_p(E)$ and discuss some topological properties and some inclusion relations. In Section 3 we characterize some matrix transformations in $\bar{\ell}_p(E)$.

2. For any sequence $x = (x_n)$, $x_n \in E$, write

$$t_{m,n}(x) = \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i}.$$

Kurtz [7] proved that $x \in \hat{c}$ (E) if and only if for some $y \in E$, $\| t_{m,n}(x) - y \| \to 0$ as $m \to \infty$ uniformly in n. Given an infinite series

$$\Sigma a_n, \; a_n \in E,$$

which we denote by a, we write

$$x_n = a_0 + a_1 + \dots + a_n.$$

We define

$$t_{-1,n}(x) = x_{n-1}$$

and write for m, n ≥ 0,

$$\varphi_{m,n}(a) = t_{m,n}(x) - t_{m-1,n}(x).$$

A straightforward calculation shows that

$$\varphi_{m,n}(a) = \frac{1}{m\,(m+1)} \sum_{i=1}^{m} i\, a_{n+i} \qquad (m \geq 1,\ n \geq 0)$$

and

$$\varphi_{o,n}(a) = a_n.$$

We define

$$\tilde{l}_p(E) = \{a : a_n \in E,\ \sum_m \|\varphi_{m,\cdot}(a)\|^p \text{ converges uniformly in } n\}$$

and

$$\hat{l}_p(E) = \{a : a_n \in E,\ \sup_n \sum_m \|\varphi_{m,n}(a)\|^p < \infty\}.$$

It is evident that if $a \in \tilde{l}_1(R)$, then $x \in \hat{c}\,(E)$.

We first prove

Theorem 1. $\tilde{l}_p(E) \subset \hat{l}_p(E) \subset l_\infty\,(E)$.

Proof. Suppose that $a \in \tilde{l}_p(E)$ and write $\varphi_{m,n}$ for $\varphi_{m,n}(a)$. We have to show that $\sum_m \|\varphi_{m,n}(a)\|^p$ is bounded. By the definition there is an integer K such that

$$\sum_{m \geq K} \|\varphi_{m,n}\|^p \leq 1 \text{ (for all n)}.$$

Therefore it follows that for m $\geq$ K,

$$\|\varphi_{m,n}\| \leq 1 \text{ (for all n)}.$$

It is now enough to show that for fixed m, $\varphi_{m,n}$ is bounded. Let $m \geq 1$ fixed. A straightforward calculation shows that

$$a_{m+n} = (m+1)\,\varphi_{m,n} - (m-1)\,\varphi_{m-1,n}.$$

Hence for any fixed m $\geq K+1$, we deduce that a_i is bounded and hence $\varphi_{m,n}$ is bounded for all m,n. This completes the proof that $\hat{\iota}_p(E) \subset \hat{\iota}_\rho(E)$. To show that $\hat{\iota}_p(E) \subset \hat{\iota}_\infty(E)$, observe that

$$\sup_{m,n} \| \varphi_{m,n} \| \geq \sup_{n} \| \varphi_{1,n} \| = \tfrac{1}{2} \| a \|_\infty \tag{1}$$

$$\sup_{m,n} \| \varphi_{m,n} \| \leq \sup_{n} \| \frac{a}{m(m+1)} \|_\infty \sum_{i=1}^{m} i = \tfrac{1}{2} \| a \|_\infty \tag{2}$$

$$\sup_{m,n} \| \varphi_{m,n} \| \leq \sup_{n} (\sum_{m} \| {}_{m,n} \|^p)^{1/p} \tag{3}$$

where $\| a \|_\infty$ denotes the usual $\iota_\infty(E)$ -norm of a. Now the result follows from the inequalities (1), (2) and (3).

REMARK. It is no use trying to generalize $\iota_\infty(E)$ to $\hat{\iota}_\infty(E)$ just as c(E) has been generalized to $\hat{c}(E)$. For if we define $\hat{\iota}_\infty(E)$ as the set of all sequences $a = (a_n)$, $a_n \in E$ for which

$$\sup_{m,n} \| \varphi_{m,n}(a) \| < \infty,$$

then it follows from inequalities (1) and (2) that $\iota_\infty(E) = \hat{\iota}_\infty(E)$

We have

THEOREM 2. (i) Let $1 < p < \infty$. Then $\hat{\iota}_p(E)$ and $\hat{\iota}_p(E)$ are Banach spaces with the norm

$$\| a \|_{\hat{p}} = \sup_{n} (\sum_{m} \| \varphi_{m,n}(a) \|^p)^{1/p}$$

(ii) Let $1 \leq p \leq q < \infty$. Then $\hat{\iota}_p(E) \subset \hat{\iota}_q(E)$ and $\hat{\iota}_p(E) \subset \hat{\iota}_q(E)$.

Proof. Let $a \in \hat{\iota}_p(E)$. Then from Theorem 1 it follows that $\| a \|_{\hat{p}}$ is well-defined. It can be proved by standard arguments that (4) defines a norm on $\hat{\iota}_p(E)$ and $\hat{\iota}_p(E)$. To show that, with the norm topology, the spaces are complete, let (a^i) be a Cauchy sequence in $\hat{\iota}_p(E)$. Then since

$$\| a^i_n - a^j_n \| = \| \varphi_{o,n}(a^i) - \varphi_{o,n}(a^j) \| \leq \| a^i - a^j \|$$

it follows that (a_n^i) is a Cauchy sequence in E and hence $a^i_n \to a_n \in E$. Put $a = (a_n)$. For every $\varepsilon > 0$ there is N such that for every K and i, $j > N$.

$$\sum_{m=o}^{K} \|\varphi_{m,n}(a^i - a^j)\|^p < \varepsilon \qquad \text{(for all } n).$$

Now taking limit as $j \to \infty$ and then as $K \to \infty$, we get for $i > N$,

$$\sum_{m} \|\varphi_{m,n}(a^i - a)\|^p \leq \varepsilon \text{ (for all } n).$$

This shows that $a^i \to a \in \mathring{\iota}_p(E)$ and completes the proof of (i). (ii) can be obtained from Hölder's inequality.

We now establish the inclusion between $\iota_p(E)$ and $\mathring{\iota}_p(E)$.

THEOREM 3 (i) Let $1 \leq p < \infty$. Then $\iota_p(E) \subset \hat{\iota}_p(E)$.

(ii) Let $0 < p \leq \frac{1}{2}$. Then $a \in \mathring{\iota}_p(E)$ implies that $a_n = 0$ $(n \geq 1)$.

To prove (i) we require the following lemma.

LEMMA A. Suppose that

(i) $\sum_m |a_{m,n}|$ convergens for each n,

(ii) $\sum_m |a_{m,n}| \to 0$ as $n \to \infty$

then

$$\sum_m |a_{m,n}| \text{ converges uniformly in } n.$$

See, for example, Maddox [13], p. 167.

Proof of Theorem 3 (i). Suppose that $a \in \iota_p(E)$. Let $m \geq 1$ and $n \geq 0$. By Hölder's inequality when $p > 1$ and trivally when $p = 1$ we have

$$\|\varphi_{m,n}(a)\|^p \leq \frac{1}{m(m+1)^p} \sum_{i=1}^{m} i^p \|a_{n+i}\|^p.$$

Hence

$$\sum_{m=1}^{\infty} \|\varphi_{m,n}(a)\|^p \leq \sum_{i=1}^{\infty} i^p \|a_{n+1}\|^p \sum_{m=i}^{\infty} \frac{1}{m(m+1)} p < \sum_{i=1}^{\infty} \|a_{n+i}\|^p$$

Since $\varphi_{o,n}(a) = a_n$ we have

$$\sum_{m} \|\varphi_{m,n}(a)\|^p \leq \sum_{i=n}^{\infty} \|a_i\|^p.$$

Thus the hypothesis of Lemma A are satisfied for $a_{m,n} = \|\varphi_{m,n}(a)\|^p$ and the result is now evident.

(ii) Let $a \in \hat{l}_p(E)$ and put $\varphi_{m,n}$for $\varphi_{m,n}(a)$. Write

$$T_{m,n} = m\,(m+1)\,\varphi_{m,n} = \sum_{i=n+1}^{n+m} (i-n)\; a_i$$

We are given that for all $n \geq 0$,

$$\sum_{m=1}^{\infty} \frac{\|T_{m,n}\|^p}{m^2p} < \infty.$$

Hence

$$\sum_{m=3}^{\infty} \frac{\|T_{m,n}-2T_{m-1,n+1}+T_{m-2,n+2}\|^p}{m^2p}$$

$$\leq \sum_{m=3}^{\infty} \frac{\|T_{m,n}\|^p}{m^2p} + 2^p \sum_{m=3}^{\infty} \frac{\|T_{m-1,n+1}\|^p}{m^2\rho} + \sum_{m=3}^{\infty} \frac{\|T_{m-2,n+2}\|^p}{m^2\rho} < \infty$$

But a straightforward calculation shows that

$$T_{m,n} - 2T_{m-1,n+1} + T_{m-2,n+2} = a_{n+1}.$$

Now since $p \leq \frac{1}{2}$, we must have $a_{n+1} = 0$. and since $n \geq 0$, this completes the proof.

We now discuss some properties of compact sets in $\hat{l}_p(E)$ and $\hat{l}_\rho(E)$.

THEOREM 4. If a set $K \subset \hat{l}_p(E)$ is compact, then

i) if f_k: $\hat{l}_p(E) \to E$ is given by $f_k(a) = a_k$ for all $a \in \hat{l}_p(E)$, then $f_k(K)$ is compact for all $k \geq 0$.

ii) given $\varepsilon > 0\ \exists\ j_o = j_o(\varepsilon)$ such that

$$\left(\sum_{m=j+1}^{\infty} \|\varphi_{m,n}(a)\|^p\right)^{1/p} < \varepsilon \quad \forall\, a \in K, \forall\, j \geq j_0.$$

In particular, the above result holds for a compact subset of $\hat{\ell}_p(E)$.

Proof. (i) Let $K \subset \hat{\ell}_p(E)$ be compact. For a, b $\in \{\hat{\ell}_p(E)$,

$$\begin{aligned} \|f_n(a) - f_n(b)\| &= \|a_n - b_n\| \\ &= \|\varphi_{o,n}(a) - \varphi_{o,n}(b)\| \\ &\leq \|a-b\|. \end{aligned}$$

Thus the coordinate functions f_n: $\hat{\ell}_p(E) \to E$ are continuous and so $f_k(K)$ is compact for all $k \geq 0$.

(ii) Let $\varepsilon > 0$ be given and for $a \in \hat{\ell}_p(E)$, let

$$U\left(a, \frac{\varepsilon}{2}\right) = \left\{ b \in \hat{\ell}_p(E) : \|b - a\| < \frac{\varepsilon}{2} \right\}.$$

Then

$$K \subset \bigcup_{a \in K} U\left(a, \frac{\varepsilon}{2}\right).$$

Since K is compact, $\exists\ a^1, a^2, \ldots, a^N \in N$ such that

$$K \subset \bigcup_{i=1}^{N} U\left(a_i, \frac{\varepsilon}{2}\right).$$

If $a \in K$, then

$$\left(\sum_m \|\varphi_{m,n}(a - a^i)\|^p\right)^{1/p} < \frac{\varepsilon}{2} \quad (\forall n)$$

For each i, $\exists\ j_i$ such that

$$\left(\sum_{m=j+1}^{\infty} \|\varphi_{m,n}(a^i)\|^p\right)^{1/p} < \frac{\varepsilon}{2} \quad (\forall j > j_i, \forall n)$$

Let $j_c = \max\limits_{1 \leq i \leq N} j_i$. Then

$$\left(\sum_{m=j+1}^{\infty} \|\varphi_{m,n}(a^{i})\|^{p}\right)^{1/p} < \frac{\varepsilon}{2} \;(\forall\, j \geq j_{o},\ 1 \leq i \leq N)$$

since $a \in K$, $\exists\, i_o$ $(1 \leq i_o \leq N)$ such that

$$\left(\sum_{m} \|\varphi_{m,n}(a-a^{i_o})\|^{p}\right)^{1/p} < \frac{\varepsilon}{2}.$$

Let $j \geq j_o$. Then

$$\left(\sum_{m=j+1}^{\infty} \|\varphi_{m,n}(a)\|^{p}\right)^{1/p}$$

$$\leq \left(\sum_{m=j+1}^{\infty} \|\varphi_{m,n}(a-a^{i_o})\|^{p}\right)^{1/p} + \left(\sum_{m=j+1}^{\infty} \|\varphi_{m,n}(a^{i_o})\|^{p}\right)^{1/p}$$

$$\leq \left(\sum_{m} \|\varphi_{m,n}(a-a^{i_o})\|^{p}\right)^{1/p} + \left(\sum_{m=j+1}^{\infty} \|\varphi_{m,n}(a^{i_o}\|^{p})\right)^{1/p}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This completes the proof.

3 . We now discuss some results on matrix transformations. Let X and Y be any two Banach sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers. We write $Ax = A_n(x)$ if $A_n(x) = \Sigma_k\, a_{nk}x_k$ converges for each n and $x_k \in E$ for each k. We denote by (X,Y) the set of matrices A such that $A_x \in Y$ for every $x \in X$.

The class (c(E), c(E)) has been characterised by Maddox [11]. Kurtz [8] characterized the classes (c(E), c(E)), (c(E), ĉ(E)) and (ĉE), c(E)) but he considered the matrices to be the elements of B [X, Y], the space of all bounded linear transformations from X into Y.

In this section we characterize the classes $(l_\infty(E), c(E))$, $(l_1(E), l_p(E))$ $(\iota_1(E),\ \hat{l}_p(E))$. We write

$$t_{m,n}(A_x) = \frac{1}{m+1} \sum_{i=o}^{m} A_{n+i}(x) = \sum_{k} a\,(n,k,m)\, x_k$$

where

$$a(n,k,m) = \frac{1}{m+1} \sum_{i=0}^{m} a_{n+i},k,$$

and

$$\varphi_{m,n}(A_a) = \frac{1}{m(m+1)} \sum_{i=0}^{m} i\ A_{n+i}(a)$$

$$= \sum_k b(n,k,m)\ a_k$$

where

$$b(n,k,m) = \frac{1}{m(m+1)} \sum_{i=1}^{m} i\ a_{n+k},\ _k\ (m \geq 1),$$

$$b(n,k,0) = a_{nk}.$$

The proof of the following theorems use standard techniques and there are nothing essentially new. But for the sake of completeness we give a short proof. The next result is the familiar Schur Theorem when E is the complex plane.

THEOREM 5. (a) A $(\iota_\infty(E), c(E))$ if and only if

i) $\sum_k |a_{nk}|$ converges uniformly in n,

ii) $\lim_{n \to \infty} a_{nk} = \alpha_k$.

(b) A $(\iota_\infty(E), \hat{c}(E))$ if any only if

i) $\sup_n \sum_k |a_{nk}| < \infty$,

ii) $\lim_{m \to \infty} a(n,k,m) = \alpha_k$ uniformly in n,

iii) $\lim_{m \to \infty} \sum_k |a(n,k,m) - \alpha_k| = 0$ uniformly in n.

Proof. We only prove the necessity of (i) of (a). Since $\sum_k a_{nk}x_k$ converges for $x \in \iota_\infty(E)$, it converges for $x \in c(E)$ and so by Maddox [11, Thorem 2] it follows that $\sup_n \sum_k |a_{nk}| < \infty$. Write $b_{nk} =$

$a_{nk}-\alpha_k$. Since $\sum_k |\alpha_k| < \infty$ it follows that $\sum_k b_{nk}x_k$ converges for $x \in \iota_\infty$ (E) and and note that (see Maddox [13], p. 169) $\sum_k |b_{nk}| \rightarrow 0$ as $n \rightarrow \infty$. Therefore by Lemma A, $\sum_k |b_{nk}|$ converges uniformly in n and this completes the proof.

THEOREM 6.Let $1 \leq p < \infty$. Then

a) A $(\iota_1(E), \iota_p(E))$ if and only if

$$\sup_n \sum_k |a_{nk}|^p < \infty,$$

b) A $(\iota_1(E), \hat{\iota}_p(E))$ if and only if

$$\sup_{n,m} \sum_k |b(n,k,m)|^p < \infty.$$

Proof. We only prove (a). Let $a \in \iota_1(E)$. Then by Minkowski's inequality, we have

$$(\sum_n \| \sum_n a_{nk}a_k \|^p)^{1/p} \leq \sum_k (\sum_k \|a_{nk}a_k\|^p)^{1/p}$$

$$= \sum_k \|a_k\| (\sum_n |a_{nk}|^p)^{1/p}$$

This proves sufficiency. For the necessity suppose that $A \in (\iota_1(E), \iota_p(E))$ and let

$$f_n(a) = (\sum_{i=1}^n \|A_i(a)\|^p)^{1/p}$$

and note that f_n is a continuous seminorm on $\iota_1(E)$ such that $\sup_n f_n(a) < \infty$. Hence the resultfollo ws by an application of Banach-Steinhaus theorem.

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