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ON CERTAIN FUNCTIONAL EQUATION HAVING SOLUTION IN THE SPACES $\Gamma_{(p,q)}(\rho)$ AND $\Gamma_{(p,q)}(\rho,T)$ OF ENTIRE FUNCTIONS

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ON CERTAIN FUNCTIONAL EQUATION HAVING SOLUTION IN THE SPACES $\Gamma_{(p,q)}(\rho)$ AND $\Gamma_{(p,q)}(\rho,T)$ OF ENTIRE FUNCTIONS

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ABSTRACT:

Using functional analysis techniques, it is shown that the functional equation

$$f(z+w_1) - \beta f(z) = g(z)$$

always has a solution in the spaces $\Gamma_{(p,q)}(\rho)$ and $\Gamma_{(p,q)}(\rho,T)$ to which g belongs. It is also shown that these spaces are Montel. The results of this paper generalize the corresponding results of Whittaker [10], Scott [8] and Krishnamurthy [5].

1. Whittaker's [10] classical theorem states that for any entire function g of order ρ there exists an entire function f of the same order such that the equation

$$(1.1) \qquad f(z+w) - f(z) = g(z)$$

is satisfied for all complex number z, where w stands for any fixed non-zero complex number. This results is further improved and extended by Scott [8] to the case of entire functions of order ρ and type T. Later on, Krishnamurthy [5], using functional analysis techniques, generalizes this result for the spaces $\Gamma(\rho,T)$, $\Gamma(\rho)$ and others where $\Gamma(\rho,T)$ denotes the space of all entire functions having growth $\{\rho.T\}$ and $\Gamma(\rho)$ represents the space of all entire functions of order not exceeding ρ. Recently, Juneja and Srivastava [2, 9] studied the spaces of entire functions of (p,q) order ρ as well as of (p,q) growth (p,T), in detail, which generalize the spaces $\Gamma(\rho)$ and $\Gamma(\rho,T)$ studied by Krishnamurthy. It is, therefore, natural to study the functional equation (1.1), in a more general form, in these new spaces. This is the purpose of the present paper which is in continuation of our previous work [2, 9].

2 . This section deals with a brief introduction of the spaces $\Gamma_{(p,q)}(\rho)$ and $\Gamma_{(p,q)}(\rho,T)$ studied by Juneja and Srivastava [2, 9].

Let $(\Gamma_{(p,q)}(\rho),d)$ represents the space of all entire functions (including constants) whose index pair does not exceed (p.q) and whose (p,q) order does not exceed ρ if of index pair (p,q), where d is the metric topology defined on $\Gamma_{(p,q)}(\rho)$ which is generated by the family of norms $\{\|f:\rho + \delta\|, \delta>0)\}$. Any element $f(z) = \sum_n a_n z^n \in \Gamma_{(p,q)}(\rho)$ is characterized by the Equation

(2 .1) $\limsup_{r\to\infty} \{(\log^{[p]} M(r, f)) / \log^{[q]} r\} \leq \rho$ or equivalently

(2 .2) $|a_n|^{1/n} \exp^{[q-1]} (\log^{[p-2]}\lambda_n)^{1/(\rho+\delta-A)} \to 0$ as $n \to \infty$ for every $\delta>0$,

$$\text{where} \quad A = \begin{cases} 1 & \text{for } (p,q) = (2,2) \\ 0 & \text{otherwise} \end{cases}$$

$$M(r,f) = \max_{|z| = r} |f(z)|$$

The norm $\|f; \rho + \delta\|$ on it is defined as

(2.3) $\|f; \rho + \delta\| = \Sigma_n |a_n| \exp (n \exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/(\rho+\delta-A)})$
where for $m = 0, 1, 2, \ldots$

$$\exp^{[m]} x = \exp (\exp^{[m-1]} x), \exp^{[-m]} x = \log^{[m]} x, \log^{[0]} x = x$$

$$\text{and } \lambda_n = \begin{cases} N_0 & \text{for } 0 \leq n \leq N_0 \\ n & \text{for } n > N_0 \end{cases} ; \; N_0 = [\exp^{[p-3]} 1] + 1$$

(Note — $\sum_n$ stands for $\sum_{n=0}^{\infty}$ throughout. For the definitions of index pair, (p,q) order, (p,q) growth etc., see [3, 4]).

Let $(\Gamma_{(p,q)}(\rho,T), d^o)$ represents the space of all entire functions (including constants) which are either of index pair less than (p,q) or are of (p,q) growth $\{\rho,T\}$, where d^o is the metric topology defined on $\Gamma_{(p,q)}(\rho,T)$ which is generated by the family of norms $\{\|f, \rho, T + \delta\|, \delta > 0\}$. Any element $f(z) = \Sigma_n a_n z^n \in \Gamma_{(p,q)}(\rho,T)$ is characterized by the equation

(2.4) $\limsup_{r\to\infty} \{(\log^{[p-1]} M(r,f) / (\log^{[q-1]} r)^{\rho}\} \leq T$ or equivelently

(2.5) $|a_n|^{1/n} \exp^{[q-1]} \left(\frac{M_1}{T+\delta} \log^{[p-2]}\lambda_n\right)^{1/(\rho-A)} \to 0$ as $n\to\infty$ for every $\delta>0$

(2.6) $M_1 \equiv M_1(p,q) = \begin{cases} 1 & \text{if } p \geq 3 \\ 1/e\rho & \text{if } (p,q) = (2,1) \\ \dfrac{(\rho-1)^{\rho-1}}{\rho^{\rho}} & \text{if } (p,q) = (2,2) \end{cases}$

The norm $\|f, \rho, T+\delta\|$ on it is defined as

(2.7) $\|f, \rho, T+\delta\| = \sum_n |a_n| \exp\left(n \exp^{[q-2]} \left(\frac{M_1}{T+\delta} \log^{[p-2]}\lambda_n\right)^{1/(\rho-A)}\right)$

where λ_n and A are defined as above.

Characterization of continuous linear functionals and the convergence criteria in these spaces have also been obtained [2, 9], In fact, it is shown that

Theorem 2.1 (a) Every continuous linear functional Ψ defined on $\Gamma_{(p,q)}(\rho)$ is of the form $\Psi(f) = \sum_n c_n a_n$, $f(z) \sum_n a_n z^n \in \Gamma_{(p,q)}(\rho)$ where

(2.8) $\limsup_{n\to\infty} |c_n|^{1/n} \exp\{-\exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/\rho-\delta+A}\} < 1$

for some $\delta > 0$, and conversely.

(b) Every continuous linear functional Ψ defined on $\Gamma_{(p,q)}(\rho,T)$ is of the form $\Psi(f) = \sum_n c_n a_n$, $f(z) = \sum_n a_n z^n \in \Gamma_{(p,q)}(\rho,T)$ where

(2.9) $\limsup_{n\to\infty} \frac{(\log^{[q-1]} |c_n|^{1/n})^{(\rho-A)}}{\log^{[p-2]}\lambda_n} < \frac{M_1}{T}$, and conversely.

Theorem 2.2 Convergence in $(\Gamma_{(p,q)}(\rho),d)$ and $(\Gamma_{(p,q)}(\rho,T), d^{o})$ are equivalent to uniform convergence over compact subset of

$D_a = \{z : |z| > a\}$ relative to the function

$\exp\left(\int_a^{|z|} \frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t} dt\right)$ and

$\exp\left(\int_a^{|z|} \frac{\exp^{[p-2]}((T+\delta) \log^{[q-1]}t)^{\rho}}{t} dt\right)$ respectively

for each $\delta > 0$ where $a = \max(1, \exp^{[q-2]} 1)$

Theorem 2.3 Convergence in $(\Gamma_{(p,q)}(\rho),d)$ and $(\Gamma_{(p,q)}(\rho,T), d^o)$ are equivalent to the convergence in normed spaces $(\Gamma_{(p,q)}(\rho), \|\,.,\, \rho+\delta\,\|)$ and $(\Gamma_{(p,q)}(\rho,T), \|.,\, \rho,\, T+\delta\,\|)$ respectively for each $\delta>0$.

Now we state few well known results.

Lemma 2.1 [7; pp. 22]: The following two properties of a set E in a topological vector space are equivalent: (a) E is bounded (b) If $\{x_n\}$ is a sequence in E and $\{t_n\}$ is a sequence of complex number Ψ such that $t_n \to 0$ as $n \to \infty$, then $t_n x_n \to 0$ as $n \to \infty$.

Lemma 2.2 [7; pp 68]: In a locally convex space X, every weakly bounded set is strongly bounded.

Lemma 2.3 [6; pp. 41]: A subset X_2 of a complete metric space X is relatively compact if and only if X contains finite ε-net for the set X_2 for arbitrary $\varepsilon > 0$.

3. In this section, we prove that the spaces $(\Gamma_{(p,q)}(\rho),d)$ and $(\Gamma_{(p,q)}(\rho,T),d^o)$ are Montel. First we prove

Theorem 3.1 Let $E \subset \Gamma_{(p,q)}(\rho)$ and $f(z) = \Sigma_n a_n z^n$ be an arbitrary element in E. Then E is bounded if and only if

(3.1) the sequence $\{a_n\}$ is bounded, uniformly for all $f \in E$, and

(3.2) given $\varepsilon > 0$, whatever may be $f \in E$, for each $\delta > 0$, there exists $n_0(\varepsilon, \delta)$ such that

$$|a_n|^{1/n} \exp^{[q-1]} (\log^{[p-2]}\lambda_n)^{1/(\rho+\delta-A)} \leq \varepsilon \text{ for } n \geq n_0.$$

Proof (Sufficient Part) In virtue of Lemmas 2.1 and 2.2, it is sufficient to show that if $f_p(z) = \sum_n a_n(p)\, z^n$ is an arbitrary sequence in E and $\{t_p\}$ is a sequence of complex number such that $t_p \to 0$, then $\Psi(t_p f_p) \to 0$ as $p \to \infty$ for all continuous linear functional Ψ on $\Gamma_{(p,q)}(\rho)$. Because of Theorem 2.1 (a), $\Psi(t_p f_p) = \sum_n t_p a_n(p)\, c_n$ where $\{c_n\}$ satisfied (2.8).

By (2.8), given $\eta > 1$ there exitss $n_1(\eta)$ such that for some $\delta = \delta_1$

$$(3.3) \quad |c_n|^{1/n} \exp\, \{-\exp^{[q-2]}(\log^{[p-2]}\lambda_n)^{1/(\rho+\delta_1-A)}\} \leq \frac{1}{\eta} \text{ for } n \geq n_1$$

However, by (3.2), given ε $(0 < \varepsilon < \eta)$ and $\delta = \delta_1$, there exists $n_0(\varepsilon, \delta_1)$, independent of p, such that

(3.4) $|a_n^{(p)}|^{1/n}\exp^{[q-1]}(\mathrm{llog}^{[p-2]}\lambda_n)^{1/(\rho+\delta_1-A)} \leq \varepsilon$ for $n \geq n_0$.

Choose $N = \max(n_0, n_1)$. In virture of Eq. (3.1), (3.3) and (3.4), it follows that $\sum_n |a_n^{(p)} c_n|$ is bounded, the bound being independent of p. Thus $|\Psi(t_p f_p)| \leq \eta_1 |t_p| \to 0$ as $p \to \infty$ for every Ψ. So E is bounded.

(Necessary Part) Suppose E is bounded in $(\Gamma_{(p,q)}(\rho), d)$ so for every $\delta > 0$, the norm $\|f, \rho + \delta\|$ is bounded because of the result [7, Theorem 1.37 pp. 26] where $f \in E$. So fixing δ, we have $|a_0| + \sum_{n=1}^{\infty} |a_n| \exp(n \exp^{[q-2]}(\log^{[p-2]}\lambda_n)^{1/(\rho+\delta-A)}) \leq \eta_2$ for all $f(z) = \sum_n a_n z^n \in E$. This immediately implies that $\{a_n\}$ is uniformly bounded for all $f \in E$.

Now, suppose (3.2) fails to hold. Then, for a given $\varepsilon > 0$ and some δ_0, these exists a sequence $\{f_p\}_{p=1}^{\infty}$ of E,

$f_p(z) = \sum_n a_n^{(p)} z^n$ and a corresponding sequence of positive integers $n_1, n_2, \ldots (n_1 < n_2 < \ldots)$ such that

(3.5) $|a_{n_p}^{(p)}|^{1/n_p} \exp^{[q-1]}(\log^{[p-2]}\lambda_{n_p})^{1/(\rho+\delta_0-A)} > \varepsilon$, $p = 1,2,\ldots$ clearly
(3.6) $p \leq n_p$. Define

$$(3.7)\quad c_n = \begin{cases} 0 & \text{for } n \neq n_1, n_2 \ldots \\ \dfrac{p}{|a_n^{(p)}| \,\mathrm{sgn}\,(a_n^{(p)})} & \text{for } n = n_p, \; p = 1,2 \ldots \end{cases}$$

Consider, for $\delta < \delta_0$

$$\limsup_{n\to\infty} \frac{|c_n|^{1/n}}{\exp^{[q-1]}(\log^{[p-2]}\lambda_n)^{1/(\rho+\delta-A)}}$$ which is zero because of

(3.5), (3.6) and (3.7). This implies, because of Theorem 2.1 (a), that Ψ defined by $\Psi(f) = \sum_n c_n a_n$ is a continuous linear functional defined on $\Gamma_{(p,q)}(\rho)$. Choose $t_p = \dfrac{1}{p}$, so it goes to zero as $p \to \infty$ but

$$\Psi(t_p f_p) = \sum_n t_p c_n a_n^{(p)} = t_p \sum_n c_n a_n^{(p)} \geq t_p c_{n_p} a_n^{(p)} = 1$$

does not tend to zero as $p \to \infty$. This implies E is not weakly bounded and so not strongly bounded. Hence a contradiction to the hypothesis completes the proof.

Remark 3.1 The corresponding theorem for the space $\Gamma_{(p,q)}(\rho,T)$ can be obtained of we replace the condition (3.2) by.

(3.8) Given $\varepsilon > 0$, whatever may be $f \in E \subset \Gamma_{(p,q)}(\rho,T)$, for each $\delta > 0$, there exists n_0 (ε, δ) such that

$$|a_n|^{1/n} \exp^{[q-1]} \left(\frac{M_1}{T+\delta} \log^{[p-2]} \lambda_n \right)^{1/(\rho-A)} \leq \varepsilon \text{ for } n \geq n_0.$$

The proof runs on the same lines.

Lemma 3.1 (a) Let E be a bounded set in $(\Gamma_{(p,q)}(\rho),d)$, then given $\varepsilon > 0$ there exists, for each $\delta > 0$, an n_2 (ε, δ) such that for whatever may be $f(z) = \sum_n a_n z^n \in E \subset \Gamma_{(p,q)}(\rho)$

$$\left\| \sum_{n=n_2}^{\infty} a_n z^n, \quad \rho + \delta \right\| < \varepsilon$$

(b) Let E be a bounded set in $(\Gamma_{(p,q)}(\rho.T), d^o)$, then given $\varepsilon > 0$ there exists, for each $\delta > 0$, an n_2 (ε, δ) such that for whatever may be $f(z) = \sum_n a_n z^n \in E$

$$\left\| \sum_{n=n_2}^{\infty} a_n z^n, \rho, T + \delta \right\| < \varepsilon.$$

The proof follows from Theorem 3.1 so we omit it.

Now, we have main theorem of this section.

Theorem 2.2 The spaces $(\Gamma_{(p,q)}(\rho), d)$ and $(\Gamma_{(p,q)}(\rho,T), d^o)$ are Montel spaces. In other words, they are barrelled spaces in which every bounded set is relatively compact.

Proof. Since these spaces are Frechet so are barrelled. It is now remained to show that every bounded set E in these spaces is relatively compact. But by Lemma 2,3, it is enough to show that these spaces contain, for arbitrary $\varepsilon > 0$, a finite ε-net for the subset E of the space in question.

For this, assume E is a bounded subset of the space in question and α be a metric on E. Let $f = \sum_n a_n e_n \in E$ where $e_n(z) = z^n$ for $n = 0,1,2,\ldots$, Define $S = \{f_1 = \sum_{n=0}^{n_0-1} a_n e_n$ such that $\alpha\left(\sum_{n=n_0}^{\infty} a_n e_n, 0\right) < \varepsilon/2\}$.

This is possible because of the Lemma 3.1 and the Theorem 2.3. Clearly S is finite dimensional set with bases $e_0, e_1, \ldots, e_{n_0-1}$ and also bounded. So S is compact. Therefore there exists an $\frac{\varepsilon}{2}$ net in S which is obviously an ε-net for the whole of E, because, if $f = \sum_n a_n e_n \in E$ and $f_1 = \sum_{n=0}^{n_0-1} a_n e_n \in S$ then for some g in the $\frac{\varepsilon}{2}$ net for S, we have

$\alpha(f_1 - g, 0) < \varepsilon/2$. So

$\alpha(f - g, 0) \leq \alpha(f - f_1, 0) + \alpha(f_1 - g, 0) < \varepsilon$.

This completes the proof.

4. In this section we give few lemmas which are used in the final section. First ve have

Lemma 4.1 If B is a continuous linear endomorphism of any one of the spaces $(\Gamma_{(p,q)}(\rho), d)$ and $(\Gamma_{(p,q)}(\rho,T), d^o)$, then $U = B - \beta I$, where β is any nonzero complex number and I is the identity transformation, maps bounded closed sets onto closed stts.

Proof. Let K denote any one of the space under consideration and suppose E is a bounded closed set in K. For $f_n \in E$, $n = 1,2,3 \ldots$, let $\lim_{n\to\infty} U(f_n) = g_o$. Since B is continuous and the spaces in question are Montel so it maps bounded set $\{f_n\}$ into a relatively compact set $\{B(f_n)\}$. Hence there must exist a subsequence $\{B(f_{ni})\}$, say, which converges to an element $h_o \in K$ (say). Since $\beta f_{ni} = B(f_{ni}) - U(f_{ni})$, it follows that

$\lim_{i\to\infty} f_{ni} = \frac{1}{\beta}(h_o - g_o) \in E$ as E is closed. Thus $U\left(\frac{h_o - g_o}{\lambda}\right) = \lim_{i\to\infty} U(f_{ni}) = g_o$. Hence the lemma.

Using Lemma 4.1, we can easily prove the following Lemma on the same lines as adopted in [1, Theorem 5, pp, 489].

Lemma 4.2 The operator $U = B - \mu I$, where B, β and I have the same meaning as in Lemma 4.1, has a closed range and so is an onto mapping whenever the range is also dense in the space in question.

Lemma 4.3 Let $\varnothing_1$ and Ψ_1 be two positive indefinitely increasing functions such that $\varnothing_1(x) / \Psi_1(x) \to 0$ as $x \to \infty$, then for m = 1,2, ...; $(\exp^{[m]} \varnothing_1(x) - \exp^{[m]} \Psi_1(x)) \to -\infty$ as $x \to \infty$. The proof is straight forward, hence omitted.

5. (Throughout this section, let K stands for any one of the spaces $(\Gamma_{(p,q)}(\rho),d)$ and $(\Gamma_{(p,q)}(\rho,T),d^o))$.

In this section, we consider the functional equation

(5.1) $f(z+w_1) - \beta f(z) = g(z)$

where w_1 and β are any nonzero complex numbers and the entire function $g \in K$.

For $f \in K$, define

(5.2) $(B_1(f))(z) = f(z+w_1)$, $z \in C$.

Obviously, B_1 is linear. By equations (2.1) and (2.4), it follows that B_1 is an endomorphism of K.

We now establish

Theorem 5.1 The operator B_1 defined by (5.2) is continuous in the topology of K.

Proof. (For the space $\Gamma_{(p,q)}(\rho)$): Let $f_n \to 0$ in $(\Gamma_{(p,q)}(\rho),d)$ Then, by Theorem 2.2

$$(5.3)\quad |f_n(z+w_1)| \exp\left\{-\int_a^{|z+w_1|} \frac{\exp^{[p-2]}(\log^{[q-1]} t)^{\rho+\delta}}{t}\, dt\right\} \to 0$$

as $n \to \infty$ uniformly in D_a, for each $\delta > 0$. To show that B_1 is continuous, we have to prove that

$$(5.4)\quad |f_n(z+w_1)| \exp\left\{-\int_a^{|z|} \frac{\exp^{[p-2]}(\log^{[q-1]} t)^{\rho+\delta'}}{t}\, dt\right\} \to o \text{ as}$$

$n \to \infty$ uniformly in D_a, for each $\delta' > 0$. Thus, in order that (5.3) may imply (5.4), we need only to show that for each $\delta < \delta'$

$$(5.5)\quad I_0 \equiv \exp\left\{ \int_a^{|z+w_1|} \frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}\,dt - \int_a^{|z|} \frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta'}}{t}\,dt \right\}$$

is bounded uniformly in D_a. Clearly,

$$I_0 \leq \exp\left\{ \int_a^{(|z|+|w_1|)} \frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}\,dt - J_1 \right\}$$

where $J_1 \equiv \displaystyle\int_a^{|z|} \frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta'}}{t}\,dt.$

Thus

$$I_0 \leq \exp \int_{a-|w_1|}^{|z|} \frac{\exp^{[p-2]}(\log^{[q-1]}(t+|w_1|))^{\rho+\delta}}{t}\,dt - J_1 \Big\}$$

$$= \exp \int_{a-|w_1|}^{a} \frac{\exp^{[p-2]}(\log^{[q-1]}(t+|w|))^{\rho+\delta}}{t}\,dt$$

$$+ \int_a^{|z|} \left[\frac{\exp^{[p-2]}(\log^{[q-1]}(t+|w_1|))^{\rho+\delta}}{t} - \frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta'}}{t}\right] dt \Big\}.$$

Or

$$(5.6)\quad I_0 \leq \exp\left\{ \eta + \int_a^{|z|=r} \frac{J^o_1(t)}{t}\,dt \right\},$$

where η being a constant and

$$J_1^o(t) \equiv \{\exp^{[p-2]}(\log^{[q-1]}(t+|w_1|))^{\rho+\delta} - \exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta'}\}$$

Let $\varnothing_1(r) = (\log^{[q-1]}(r+|w_1|))^{\rho+\delta}$ and

$\Psi_1(r) = (\log^{[q-1]}r)^{\rho+\delta'}$. Clearly $\dfrac{\varnothing_1(r)}{\Psi_1(r)} \to 0$

as $r \to \infty$, so by Lemma 4.3, $J_1^o(r) \to -\infty$ as $r \to \infty$. Hence, howsoever large η_1 (> 0) may be, there exists r_0 such that for $r \geq r_0$, $J_1^o(r) \leq -\eta_1$. Therefore, by (5.6)

$$I_0 \leq \exp\left\{\eta + \eta_2 - \eta_1 \int_{r_0}^{r} \frac{dt}{t}\right\}, \quad \eta_2 = \text{constant}$$

$$= \exp\left\{\eta + \eta_2 - \eta_1 \log \frac{r}{r_0}\right\} = 0\ (1), \text{ uniformly in } D_\alpha.$$

This completes the proof.

The proof of Theorem 5.1 for $\Gamma_{(p,q)}(\rho,T)$ is similar and hence omitted.

Next we have

Lemma 5.1 Let U_1, defined by $U_1 = B_1 - \beta I$, be an operator from K to K. Then the range of U_1 is dense in K.

Proof. Since $\left\{e_n\right\}_{n=0}^{\infty}$, $e_n(z) = z^n$, is a basis in K so any element $f \in K$ can be expressed as $f = \sum_n a_n e_n$. Now

$$(U_1(e_n))(z) = (z+w_1)^n - \beta z^n = \alpha_n \text{ (say)}.$$

The elements e_0, e_1, e_2 .... can all be represented as finite lineear combinations of $\{\alpha_n\}$ and so every element $f \in K$ can be uniquely written as

$f = \sum_n a_n' \alpha_n$. So $f = \sum_n a'_n U_1(e_n) = \lim_{p\to\infty} U_1\left(\sum_{n=0}^{p} a'_n e_n\right)$ which shows that $U_1(K)$ is dense in K.

Finally, we have

Theorem 5.2 For every $g \in K$, there exists an $f \in K$ satisfying

$$f(z+w_1) - \beta f(z) = g(z)$$

where w_1 and β are any nonzero complex numbers.

Proof. Theorem 5.1, Lemma 5.1 and Lemma 4.2 give that the mapping $U_1 = B_1 - \beta I$ is onto. So for every $g \in K$ there exists f in K such that

$$U_1(f) = g \Rightarrow ((B_1 - \beta I)f)(z) = g(z) \quad \text{for every } z \in C$$

$$\Rightarrow f(z+w_1) - \beta f(z) = g(z).$$

Hence the theorem.

Remarks 5.1 It is clear that if the entire function g in (5.1) is of (p q) -growth {ρ,T} then the solution f of Equation (5.1) must also be of (p,q) growth (ρ,T). Similar remarks applies if $g \in \Gamma_{(p,q)}(\rho)$.

For p = 2 and q = 1 the functional Equation (5.1) has been established by Krishnamurthy [5]. Also for β = 1, p = 2 and q = 1 we get results of Whittaker [10] and Scott [8].

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