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ON THE SPACE D_p^A OF ANALYTIC FUNCTIONS

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ON THE SPACE D_p^Λ OF ANALYTIC FUNCTIONS

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ABSTRACT

Let D_p^Λ $(1 \leq p < \infty)$ be the class of functions f, $f(z) = \sum_n a_n z^n$ such that $\sum_n |a_n v_n|^p < \infty$, where $\{v_m\}_{n=0}^{\infty}$ is any fixed sequence of nonzero complex numbers satisfying $\lim\inf |v_n|^{1/n} = r$ $(0 < r \leq \infty)$. D_p^Λ under suitable topological and algebraic structures forms a Banach algebra without identify. The properties pertaining to quasi-invertible elements, topological zero divisors etc. have been studied. Multipliers and matrix transformations between D_p^Λ and various other known Frèchet spaces have been discussed.

1. Let $\{v_n\}_{n=o}^{\infty}$ be any fixed sequence of non-zero complex numbers satisfying

$$\lim_{n \to \infty} \inf |v_n|^{1/n} = r \qquad (0 < r \leq \infty)$$

Detine a function $\Lambda : C \to C$ by $\Lambda(z) = \sum_n \frac{z^n}{v_n}$. Obviously, Λ is analytic in the disc $E_r = \{z : |z| < r\}$. We consider the set $D_p^\Lambda = \{f : f(z) = \sum_n a_n z^n \text{ such that } \sum_n |a_n v_n|^p < \infty, 1 \leq p < \infty\}$.

It is easily seen that D_p^Λ forms a vector space with respect to usual point wise scalar multiplication and pointwise addition.

Define, for $f,g \in D_p^\Lambda$, $f(z) = \sum_n a_n z^n$, $g(z) = \sum_n b_n z^n$

(1.1) $\|f\| = (\sum_n |a_n v_n|^p)^{1/p}$

and

(1.2) $(f * g)(z) = \sum_n a_n b_n v_n z^n$

[Through out this paper summation without limits runs from 0 to ∞] Clearly, $(D_p^\Lambda, \| . \|)$ is a Banach space and $(D_p^\Lambda, \| . \|, *)$ is a Branch algebra without identity. The space D_p^Λ was introduced and discussed by Srivastava et. al. [1983] and subsequently Nanda et. al. [1984] obtained some more interesting results for D_p^Λ. In continuation, we obtain some further results for it in this paper. In Section 1 we characterize quasi–invertible elements and topological zero divisors. Sections 2 and 3 deal with multipliers, point wise multipliers and matrix transformations. Some results of this paper generalise the corresponding results of Somasundaram [1974], Agarwal and Srivastava [1983]. We have

THEOREM 1.1. The set of all two sided topological divisors of zero of D_p^Λ is the set D_p^Λ itself.

Proof. Let $f(z) = \sum_n a_n z^n$ be an arbitrary element of D_p^Λ.

Consider the sequence $\{g_n\}$, $g_n(z) = \dfrac{z^n}{v_n}$. Obviously $g_n \in D_p^\Lambda$ and $\|g_n\| = 1$, for every $n \geq 0$. Further

$$(f * g_n)(z) = (g_n * f)(z) = a_n z^n$$

Hence

$$\| f * g_n \| = \| g_n * f \| = | a_n v_n | \to 0 \quad \text{as } n \to \infty.$$

Hence the theorem.

THEOREM 1.2. An element $f(z) = \sum_n a_n z^n$ of D_p^Λ is quasi–invertible if

(1.3) $\inf_n \{ | a_n v_n + 1 | \} = k > 0,$

the quasi–inverse of f(z) being the function $g(z) = \sum_n b_n z^n$ where

(1.4) $b_n = \dfrac{- a_n}{a_n v_n + 1}$

On the otherhand, if there exists n such that $a_n = -\frac{1}{v_n}$,

then f(z) is not quasi–invertible.

Proof. Suppose the inequality (1.3) holds. The function g(z) defined by (1.4) belongs to D_p^Λ. Also by (1.4) we have

$$(f + g + f * g)(z) = 0 = (f + g + g * f)(z).$$

Hence g(z) is a quasi–inverse of f(z).

Because of (1.4), if there exists n with $a_n = -\frac{1}{v_n}$, it then trivially holds that f(z) has no quasi–inverse.

2. This section is devoted to the study of multipliers.

Let X and Y be two Frèchet spaces of sequences $X = \{\{\rho_n\}\}$ and $Y = \{\{\beta_n\}\}$. A sequence $\{\lambda_n\}$ is said to be a sequential multiplier (simply "multiplier") from X to Y if $\{\lambda_n \rho_n\} \in Y$ whenever $\{\rho_n\} \in X$. Multiplier can be treated as an operator $B_\lambda : X \to Y$ defined as $B_\lambda \{\rho_n\} = \{\lambda_n \rho_n\}$. Here we study the multipliers from D_p^Λ to D_p^μ and then from H^p, the Hardy class with pth mean bounded, $0 < p < 1$, to D_p^Λ and from B^p $(0 < p < 1)$ to D_p^Λ. The Hardy class H^p is defined as

$$H^p = \left\{ f : M_p(r,f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty \text{ as } r \to 1 \right\}$$

and a larger class B^p is defined as

$$B^p = \left\{ f : \int_0^1 (1-r)^{1/p-2} M_1(r,f)\, dr < \infty \right\}.$$

We have

THEOREM 2.1. A function λ; $\lambda(z) = \sum_n \lambda_n z^n$ is a multiplier from D_p^Λ to D_p^μ if and only if

$$(2.1) \qquad \lambda_n = O\left(\left| \frac{v_n}{\mu_n} \right| \right)$$

Proof. Let $\lambda(z)$ be a multiplier from D_p^Λ to D_p^μ.
Applying closed graph theorem to the operator

$B_\lambda : D_p^\Lambda \to D_p^\mu$, we get

$$\sum_n |\lambda_n \mu_n a_n|^p \le k \sum_n |a_n v_n|^p$$

where $f(z) = \sum_n a_n z^n$ is an arbitrary member of D_p^Λ. Putting

$f = \delta_i$, $\delta_i(z) = z^i$, $i = 0, 1, \ldots$ we get

$|\mu_n \lambda_n|^p \le k \, |v_n|^p$ for all $n \ge 0$.

Hence (2.1) holds.

Let, on the otherhand (2.1) hold. Let $f(z) = \sum_n a_n z^n$ be an arbitrary member of D_p^Λ. Then

$$\|\lambda . f\|_\mu^p = \sum_n |\lambda_n a_n \mu_n|^p \le k \sum_n |a_n|^p \left|\frac{v_n}{\mu_n}\right|^p |\mu_n|^p = k \sum_n |a_n v_n|^p$$
$$< \infty \text{ since } f \in D_p^\Lambda .$$

Hence $\lambda . f \in D_p^\mu$ whenever $f \in D_p^\Lambda$.

THEOREM 2.2. (a) A necessary and sufficient condition for a sequence $\{\lambda_n\}$ to be a multiplier of H^p $(0 < p < 1)$ into D_p^Λ $(0 < p \le q < \infty)$ is that

$$\sum_{n=1}^{N} n^{q/p} \, |\lambda_n v_n|^q = O(N^q) \tag{2.2}$$

(b) If $1 \le q < \infty$, $\{\lambda_n\}$ is a multiplier of $B^p(0 < p < 1)$ into D_p^Λ if and only if (2.2) holds.

(c) If $q < p$, the condition (2.2) does not imply that $\{\lambda_n\}$ multiplies H^p into D_∞^Λ, nor does it imply that $\{\lambda_n\}$ multiplies B^p into D_p^Λ if $q < 1$.

The proof follows in similar lines as done by Duren and Shields [1970].

3. This section now deals with the pointwise multipliers. A function h, $h(z) = \sum_m c_m z^m$ is said to be a pointwise multiplier from a space X to another space Y if $h . f \in Y$ whenever $f = \sum_n a_n z^n \in X$. The

The product h.f is defined by

$$(h.f)(z) = \sum_n \left(\sum_{m=o}^{n} c_m a_{n-m} \right) z^n.$$

We have

THEOREM 3.1. If h(z) is a pointwise multiplier from D_p^Λ to D_p^μ,

$$\sum_{n=m}^{\infty} |c_{n-m}\mu_n|^p \leq k\, |v_m|^p \text{ for every } m \geq 0.$$

Proof. Applying closed graph theorem to the operator

$$B_n : D_p^\Lambda \to D_p^\mu, \text{ defined by } B_n(f) = h.f$$

we get

$$\|h.f\|_\mu \leq k^{1/p} \|f\|_\Lambda \text{ for every } f \in D_p^\Lambda, \text{ for}$$

for some constant k, i.e.,

$$(3.1) \qquad \sum_n |\mu_n|^p \left| \sum_{m=o}^{n} c_{n-m}\, a_m \right|^p \leq k \sum_n |a_n v_n|^p .$$

Let $f \equiv z^n$, then from (3.1) we get the result.

4. Let X and Y be any two nonempty subsets of the space of all complex sequences and let $A = (a_{nk})$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each n and call Ax the A transform of x. If $x = (x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, then we say that A defines a matrix transformation from X into Y and we denote it by $A: X \to Y$. By (X, Y) we mean the class of matrices A such that $A:X \to Y$. In X and Y if there are some notions of limit or sum we write (X,Y,P) to denote the subset of (X, Y) which preserves the limit or sum.

In this section we view D_p^Λ as a sequence space and present some results on matrix transformations connecting D_p^Λ. Note that

$$D_p^\Lambda = \left\{ a = (a_n) : \sum_n |a_n v_n|^p < \infty \right\}$$

and

$$D^{\Lambda}_{\infty} = \{ a = (a_n) : \sup_n |a_n v_n| < \infty \}.$$

Now we state a theorem which is required in the sequel.

THEOREM 4.1. [Maddox[1970], p. 114)]. Let X be a second category p-normed space. Suppose F is a family $\{q\}$ of lower semicontinuous seminorms q such that

$$q(x) \leq H(x) < \infty,$$

for each $x \in X$ and all $q \in F$.

Then there exists a constant H, independent of x and q such that

$$q(x) \leq H \|x\|^{1/p}$$

for all $x \in X$ and all $q \in F$.

We have

THEOREM 4.2. Let $1 \leq p < \infty$. Then $A \in (D^{\Lambda}_1, D^{\Lambda}_p)$ if and only if

$$\sup_k \sum_n |a_{n,k}|^p \left|\frac{v_n}{v_k}\right|^p < \infty. \tag{4.1}$$

Proof. Let $b = \{b_k\} \in D^{\Lambda}_1$ i.e. $\sum_k |b_k v_k| < \infty$.

Assume that the condition (4.1) holds. Now

$$\Big(\sum_n \Big|\sum_k a_{n,k} b_k v_n\Big|^p\Big)^{1/p} \leq \sum_k \Big(\sum_n |a_{n,k} b_k v_n|^p\Big)^{1/p}$$

$$\leq \Big(\sum_k |b_k v_k|\Big) \Big(\sum_n |a_{n,k}|^p \left|\frac{v_n}{v_k}\right|^p\Big)^{1/p}$$

$$\leq \Big(\sum_k |b_k v_k|\Big) \sup_k \Big(\sum_n |a_{n,k}|^p \left|\frac{v_n}{v_k}\right|^p\Big)^{1/p}$$

$$< \infty$$

Hence $A_n(b) \in D^{\Lambda}_p$. To prove the necessity, suppose $A \in (D^{\Lambda}_1, D^{\Lambda}_p)$, so that

$$\sum_i |A_i(b) v_i|^p < \infty \text{ on } D^{\Lambda}_1$$

where $A_i(b) = \sum_k a_{i,k} b_k$.

Now define

$$q_n(b) = \left(\sum_{i=1}^{n} |A_i(b) v_i|^p \right)^{1/p} \qquad (n = 1, 2, \ldots)$$

By Minkowski's inequality we see that q_n is subadditive. Thus since $q_n(\lambda b) = |\lambda| \, q_n(b)$, we have that each q_n is a seminorm on D_1^Λ.

Moreover the fact that each A_i is a bounded linear functional on D_1^Λ implies that each q_n is bounded on D_1^Λ.

Hence we have a sequence $\{q_n\}$ of continuous seminorms on D_1^Λ such that

$$\sup_n q_n(b) = \sum_i \left(|A_i(b) v_i|^p \right)^{1/p} < \infty$$

for each $b \in D_1^\Lambda$. Applying Theorem 4.1 we obtain a constant H such that

$$\sum_i |A_i(b) v_i|^p \leq H^p \| b \| \text{ on } D_1^\Lambda.$$

Putting $bv = e_k$, $k = 1, 2, \ldots$. we get (4.1).

COROLLARY 4.1. $A \in (\iota_1, D_p^\Lambda)$ if and only if

$$\sup_k \sum_n |a_{n,k} v_n|^p < \infty .$$

COROLLARY 4.2. $A \in (D_1^\Lambda, \iota_p)$ if and only if

$$\sup_k \sum_n \left| \frac{a_{n,k}}{v_k} \right|^p < \infty .$$

COROLLARY 4.3. [Maddox [1970], p. 167]. $A \in (\iota_1, \iota_p)$ if and only if

$$\sup_k \sum_n |a_{n,k}|^p < \infty .$$

THEOREM 4.3. (a) $A \in (D_1^\Lambda, D_1^\Lambda, P)$ if and only if

$$\sup_k \sum_n \left| a_{n,k} \frac{v_n}{v_k} \right| < \infty ,$$

$$\sum_n a_{n,k} \frac{v_n}{v_k} = 1 \text{ for all } k.$$

(ii) (D_1^A, D_1^A, P) is closed and convex in (D_1^A, D_1^A).

This can be obtained by standard arguments and hence we omit the proof.

THEOREM 4.4. (i) $A \in (D_\infty^A, D_\infty^A)$ if and only if

$$\sup_n \sum_k \left| a_{n,k} \frac{v_n}{v_k} \right| < \infty , \tag{4.4}$$

(ii) $A \in (D_\infty^A(p), D_\infty^A)$ if and only if

$$\sup_n \sum_k \left| a_{n,k} \frac{v_n}{v_k} \right| N^{1/p_k} < \infty$$

for every integer $N > 1$.

PROOF. (i) SUFFICIENCY. Let $b = \{b_k\} \in D_\infty^A$ and let the condition hold. We have

$$\sup_n |A_n(b)| = \sup_n \left| \sum_k a_{n,k} b_k v_n \right| \leq \sup_n \left(\sum_k \left| a_{n,k} \frac{v_n}{v_k} \right| \right) (\sup_k | b_k v_k |)$$

Therefore $\{A_n(b)\}^\infty \in D_\infty^A$ and hence $A \in (D_\infty^A, D_\infty^A)$.

The necessity can be obtained by arguing as in Theorem 4.2 and by an application of Theorem 4.1.

(ii) For sufficiency take an integer

$$N > \max (1, \sup_k |b_k v_k|^{p_k}) \text{ where } b = \{b_k\} \in D_\infty^A(p).$$

Let the condition hold. Then for every n,

$$\sup |A_n(b) v_n| \leq \sup \sum \left| a_{n,k} \frac{v_n}{v_k} \right| \sup |b_k v_k|$$

$$\leq \sup \sum \left| a_{n,k} \frac{v_n}{v_k} \right| N^{1/p_k} < \infty$$

and therefore $A \in (D_\infty^A(p), D_\infty^A)$

For the necessity suppose that $A \in (D_\infty^\Lambda(p), D_\infty^\Lambda)$, but there is an integer $N > 1$ such that

$$\sup_n \sum_k \left| a_{n,k} \frac{v_n}{v_k} \right| N^{1/p_k} = \infty$$

Then the matrix $\left(a_{n,k} \frac{v_n}{v_k} N^{1/p_k}\right) \notin (D_\infty^\Lambda, D_\infty^\Lambda)$ and so there is a $b \in D_\infty^\Lambda$ with $\|b\| = 1$ such that

$$\sum_k a_{n,k} \frac{v_n}{v_k} N^{1/p_k}$$

is not bounded. Hence although $a = (N^{1/p_k} b_k) \in D_\infty^\Lambda(p)$ the sequence $A_n(a) \notin D_\infty^\Lambda$. This contradicts the fact that $A \in (D_\infty^\Lambda(p), D_\infty^\Lambda)$ and completes the proof.

COROLLARY 4.4. $A \in (D_\infty^\Lambda, \iota_\infty)$ if and only if

$$\sup_n \sum_k \left| \frac{a_{n,k}}{v_k} \right| < \infty .$$

COROLLARY 4.5. $A \in (\iota_\infty, D_\infty^\Lambda)$ if and only if

$$\sup_n \sum_k | a_{n,k} v_n | < \infty .$$

COROLLARY 4.6. [Lascarides and Maddox [1970], p. 102].

$A \in (\iota_\infty(p), \iota_\infty)$ if and only if

$$\sup_n \sum_k | a_{n,k} | N^{1/pk} < \infty .$$

COROLLARY 4.7 [Stieglitz and Tietz [1977]].

$A \in (\iota_\infty, \iota_\infty)$ if and only if

$$\sup_n \sum_k | a_{n,k} | < \infty .$$

THEOREM 4.5. Let $1 < p < \infty$. Suppose that

$A \in (D_1^\Lambda, D_1^\Lambda) \subset (D_\infty^\Lambda, D_\infty^\Lambda)$. Then $A \in (D_p^\Lambda, D_p^\Lambda)$.

The proof uses Theorem 4.2 with $p = 1$ and Theorem 4.4 (i) and follows from a simple application of Hölder's inequality.

COROLLARY 4.8. [Maddox [1970]].

Let $1 < p < \infty$ and suppose that $A \in (\iota_\infty, \iota_\infty) \cap (\iota_1, \iota_1)$. Then $A \in (\iota_p, \iota_p)$.

We conclude this paper with one more theorem on matrix trans formations.

THEOREM 4.6. (i) $A \in (D_1^A, D_\infty^A)$ if and only if

$$\sup_{n,k} \left| a_{nk} \frac{v_n}{v_k} \right| < \infty ,$$

(ii) $A \in (D_\infty^A, D_1^A)$ if and only if

$$\sum_n \sum_k \left| a_{nk} \frac{v_n}{v_k} \right| < \infty .$$

Proof. We only prove (i) . (ii) can be proved in a similar manner. Let $b \in D_1^A$ and let the condition hold. We have

$$\sup_n \left| \sum_k a_{nk} b_k v_n \right| \leq \sup_{n,k} \left(|a_{nk}| \left| \frac{v_n}{v_k} \right| \right) \sum_k |b_k v_k| < \infty.$$

and therefore $A \in (D_1^A, D_\infty^A)$.

To prove the necessity suppose that $A \in (D_1^A, D_\infty^A)$; so that

$\sup_n | A_n(b) v_n | < \infty$ on D_1^A.

Define $q_n(b) = | A_n(b) v_n |$.

Clearly each q_n is a continuous seminorm on D_1^A and $\sup_n q_n(b) < \infty$ for each $b \in D_1^A$. Therefore by Theorem 4.1 there exists a constant H such that

$$q_n(b) \leq H \| b \| \text{ on } D_1^A .$$

Now putting $b_v = e_k$ we obtain the condition and this completes the proof of (i).

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REFERENCES

AGARWAL, R.P. and SRIVASTAVA, R.K. [1983] On the space G (λ) of entire functions. Indian J. Pure and Appl. Math., No. 6, Vol. 14, 707–713.

DUREN, P.L. [1969] On the multipliers of H^p spaces. Proc. Amer. Math. Soc. (22), 24–27.

——— [1970] Theory of H^p spaces. Academic Press, New York.

DUREN, P.L. and SHIELDS, A.L. [1970] Coefficient multipliers on H^p and B^p spaces. Pac. J. Math. 32 (1), 69–78.

LASCARIDES, G.G. and MADDOX, I.J. [1970] Matrix transformations between some classes of sequences. Proc. Camb. Phil. Soc., Vol. 68, 99–104.

MADDOX, I.J. [1970] Elements of Functional Analysis. Cambridge Univ. Press.

NANDA, S. [1983] Matrix transformations and sequence spaces. Internal Report IC/83/55, International Centre for Theoretical Physics, Trieste, Italy.

NANDA, S, SRIVASTAVA, P.D. and DUTTA, S. [1984] A note on certain paranormed function spaces. Rendiconti di Mathematica, Vol. 3.

SOMASUNDARAM, D. [1974] On a Hilbert space of entire functions. Indian J. Pure and Appl. Math. Vol. 5, 921–932.

SRIVASTAVA, P.D. [1978] A study on some aspects of certain spaces and algebras of analytic functions. Ph. D. Thesis, I.I.T., Kanpur.

SRIVASTAVA, P.D., NANDA, S. and DUTTA, S. [1983] On certain paranormed function spaces. Rendiconti di Mathematica, Vol. 3.