COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Séries A_1 : Mathematique

TOME : 34 ANNÉE : 1985

SOLIDITY AND SOME SEQUENCE SPACES

By

K. CHANDRASEKHARA RAO

and

A. ALFRED GURUSINGH

1

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications dé la Faculté dés Sciéncés
dé l'Univérsité d'Ankara

La Revue "Communications de la Faculté des Sciences de l'Université d'Ankara" est un organe de publication englobant toutes les diciplines scientifiques représentées à la Faculté des Sciences de l'Université d'Ankara.

La Revue, jusqu'à 1975 a l'exception des tomes, I, II, III etait composée de trois séries

Série A : Mathématiques, Physique et Astronomie,
Série B : Chimie,
Série C : Sciences Naturelles.

A partir de 1975 la Revue comprend sept séries:

Série A_1 : Mathématiques,
Série A_2 Physique,
Série A_3 : Astronomie,
Série B : Chimie,
Série C_1 : Géologie,
Série C_2 : Botanique,
Série C_3 : Zoologie.

A partir de 1983 les séries de C_2 Botanique et C_3 Zoologie ont été réunies sous la seule série Biologie C et les numéıos de Tome commencerons par le numéro 1.

En principe, la Revue est réservée aux mémoires originaux des membres de la Faculté des Sciences de l'Université d'Ankara. Elle accepte cependant, dans la mesure de la place disponible les communications des auteurs étrangers. Les langues Allemende, Anglaise et Française seront acceptées indifféremment. Tout article doit être accompagné d'un resumé.

Les articles soumis pour publications doivent être remis en trois exemplaires dactylographiés et ne pas dépasser 25 pages des Communications, les dessins et figures portés sur les feuilles séparées devant pouvoir être reproduits sans modifications.

Les auteurs reçoivent 25 extraits sans couverture.

l'Adresse : Dergi Yayın Sekreteri
Ankara Üniversitesi,
Fen Fakültesi,
Beşevler–Ankara
TURQUIE

SOLIDITY AND SOME SEQUENCE SPACES

K. CHANDRASEKHARA RAO and A. ALFRED GURUSINGH

(Received June 3, 1985; accepted November 6, 1985)

ABSTRACT:

In this paper we investigate the solidity (normality) of the sequence spaces C_A, l_A, m_A and Γ_A.

1. INTRODUCTION AND NOTATION

We require the following sequence spaces:

c: the space of all convergent sequences.

m: the space of all bounded sequences.

l: the space of all sequences $x = \{x_k\}$ such that

$$\sum_{k=1}^{\infty} |x_k| \text{ converges.}$$

Γ: the space of all sequences $x = \{x_k\}$ such that

$$|x_k|^{1/k} \to 0 \text{ as } k \to \infty$$

ω: the space of all sequences.

Let $A = (a_{nk})$, $(n, k = 1,2, \ldots\ldots\ldots)$ be an infinite matrix. Given a sequence $x = \{x_k\}$ we write formally

$$y_n = A_n(x) = \sum_{k=1}^{\infty} a_{nk} \cdot x_k, \quad (n = 1,2, \ldots\ldots\ldots)$$

The sequence $\{y_n\} = \{A_n(x)\}$ will be denoted by Ax or y. Let X be a sequence space and let X_A be the set of all those sequences $x = \{x_k\}$ for which $Ax \in X$.

The set of all matrices transforming X into X will be denoted by (X, X). We recall the following:

A sequence space X is called solid (or normal) if an only if

$$m\,X \subset X.$$

Any matrix in (c, c) is called a conservative matrix.

A conservative matrix which preserves the limit is said to be a Toeplitz-matrix.

2. RESULTS

PROPOSITION 1. If A is a conservative matrix, which fails to sum a bounded sequence, then c_A is not solid.

Proof.

The constant sequence $e = \{1,1, \ldots\}$ is in c_A.

By our hypothesis, there exists a bounded sequence b such that $b \notin c_A$.

That is $b.e \notin c_A$ Therefore, $m.\,c_A \not\subset c_A$ showing that c_A is not solid.

COROLLARY. If A is a Toeplitz matrix, then c_A is not solid.

PROPOSITION 2. If $A \in (l, l)$, then l_A is in general, not solid.

Proof.

Let
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

That is
$$\begin{aligned} a_{n,\,2n-1} &= 1, \ (n = 1,2, \ldots) \\ a_{n,\,2n} &= 1, \ (n = 1,2, \ldots) \\ a_{n,k} &= 0, \ \text{otherwise}. \end{aligned}$$

Then

$$\sum_{n=1}^{\infty} |a_{nk}| = 1 \text{ for each fixed } k, \text{ showing that } A \in (l,l)$$

We note that

$x \in l_A$ if and only if $\sum_{k=1}^{\infty} |x_{2k-1} + x_{2k}|$ converges.

Take $x = \{1, -1, 1, -1, \ldots\}$ so that $x \in l_A$.

Take $b = \{1, -1, 1, -1, \ldots\} = x$. Then b is in m.

Now, $y = b\,x = \{1, 1, 1, \ldots\} = e$.

For e, we have $\sum_{k=1}^{\infty} |y_{2k-1} + y_{2k}| = 2 + 2 + \ldots\ldots$

which is a divergent series.

Thus $m \, . \, l_A \subset l_A$. Hence, l_A is not solid.

PROPOSITION 3. If $A \in (m, m)$, then m_A is in general, not solid.

Proof.

Let $A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\ & \cdots & & \cdots & & \cdots & \cdots \end{pmatrix}$

In other words,

$A = (a_{nk})$ is defined by

$a_{n,2n-1} = -1$, $(n = 1,2, \ldots\ldots.)$

$a_{n,2n} \quad = 1$, $(n = 1,2, \ldots\ldots.)$

$a_{n,k} \quad = 0$, otherwise.

Then

$$\sum_{k=1}^{\infty} |a_{nk}| = 2 \text{ for each fixed } n.$$

Consequently, $A \in (m, m)$.

Note that $x \in m_A$ if and only if

$Ax = \{-x_1 + x_2, -x_3 + x_4, -x_5 + x_6, \ldots\} \in m$

We take $x = \{1, 2, 3, \ldots\ldots)$ so that

$Ax = \{1, 1, 1, \ldots\}$ and $x \in m_A$.

Take $b = \{-1, 1, -1, 1, \ldots\}$ in m.

Then $bx = \{-1, 1, -3, 4, \ldots\}$ and

$A(bx) = \{3, 7, 11, \ldots 4n-1, \ldots\} \notin m$.

Thus $m.\, m_A \not\subset m_A$.

Hence m_A is not a solid space.

It is known [1] that a matrix $A = (a_{nk})$ is in (Γ, Γ) if and only if for each positive integer q there exists $p(q) \geq q$ and a constant $M(p, q)$ such that

$$\sum_{n=o}^{\infty} \frac{|a_{nk}|\, q^n}{p^k} < M(p, q) \tag{1}$$

for $k = 0, 1, 2, \ldots\ldots$

Here we take $A = (a_{nk})$, $(n, k = 0, 1, 2, \ldots)$

The above characterisation of the class (Γ, Γ) is equivalent to the following assertion, which we state as a Lemma.

LEMMA. Let $A = (a_{nk})$, $(n, k = 0, 1, 2, \ldots)$ be an infinite matrix. In order that the matrix A is in (Γ, Γ) it is necessary and sufficient that given any $\varepsilon > 0$, there is an $M > 0$, depending on ε, such that uniformly in n and k

$$a_{nk} = O(\varepsilon^n M^k) \tag{2}$$

Proof:

Suppose that (1) holds. Since the terms in the sum on the left are all non negative, each term is less than or equal to the sum, so that (1) implies that uniformly in n and k

$$a_{nk} \cdot q^n = O(p^k) \,. \tag{3}$$

Given any $\varepsilon > 0$, choose an integer q with $q > 1/\varepsilon$ and then choose p as in (1). Since $q \geq 1/\varepsilon$ (3) gives us (2) with $M = p$.

Conversely, suppose that (2) holds. Given any positive integer q choose $\varepsilon < 1/q$, and then choose M as in (2).

Then

$$\sum_{n=o}^{\infty} |a_{nk}| \, q^n = O \, \{M^k \sum_{n=o}^{\infty} \varepsilon^n \, . \, q^n\}$$
$$= O \, (M^k)$$

Since $\sum\limits_{n=o}^{\infty} \varepsilon^n \, . \, q^n$ converges (because $\varepsilon < 1/q$), it is equal to a constant. Thus if we take $p \geq M$, then (1) holds, Hence the lemma.

PROPOSITION 4. If $A \in (\Gamma, \Gamma)$, then Γ_A is not necessarily solid.

Proof.

Take A as in Proposition 3.

Writing $\{t_m\}$ for the transform of $\{x_n\}$, so that

$t_n = -x_{2n-1} + x_{2n}$, $(n = 0, 1, 2, \ldots.)$

we can verify directly that

$|x_n|^{1/n} \to 0 \Rightarrow |t_n|^{1/n} \to 0 \; (n \to \infty)$

For, if $\eta < 1$, then $|a+b|^\eta < |a|^\eta + |b|^\eta$

so that

$$|t_n|^{1/n} < |x_{2n-1}|^{1/n} + |x_{2n}|^{1/n} \tag{4}$$

since $|x_n|^{1/n} \to 0 \; (n \to \infty)$, we have $|x_n| < 1$

for sufficiently large n. Supposing that n is large enough for

$|x_{2n-1}| < 1$, $|x_{2n}| < 1$,

$|t_n|^{1/n} < |x_{2n} -1|^{1/2n-1} + |x_{2n}|^{1/2n}$

Hence, if $|x_n|^{1/n} \to 0$, then $|t_n|^{1/n} \to 0 \; (n \to \infty)$

It is now trivial that $\{1, 1, 1, \ldots.\}$ belongs to Γ_A but $\{-1, 1, -1, 1, \ldots.\}$ does not. Here we have $b = \{-1, 1, -1, 1, \ldots.\}$ in m.

So, Γ_A is not solid.

3. REMARK

These results suggest some problems for consideration, which seem to be much harder. Take, for example, Proposition 2. Supposing that $A \in (l,l)$, then l_A is not necessarily solid. But it may be solid. If l_A

$= l$ (as happens, for example, when A is the identity transformation, though it will happen in some other cases as well), then l_A solid. Again, suppose that $l_A = \omega$.

It is easily seen that this will occur if an only if there is some k_0 such that

$$a_{nk} = 0 \text{ for } k > k_0 \text{ and all } n; \tag{5}$$

and for each fixed $k \leq k_0$, we have $\{a_{nk}\} \in l$ (6)

In this case also l_A is solid.

It seems that there is a plausible conjecture that these are the only cases. In other words we have the following conjecture.

CONJECTURE 1. Let $A \in (l,l)$. Then l_A is solid if and only if either $l_A = l$ or $l_A = \omega$.

Analogous remarks apply to other propositions. There is, however one difference. In the case in which A gives the identity transformation, $l_A = l$, which is solid. But $c_A = c$, which is not solid. So, the corresponding conjecture in the case of Proposition 1 would be the following.

CONJECTURE 2. Let $A \in (c, c)$. Then c_A is solid if and only if $c_A = \omega$.

ADDENTUM.

Take $A = (a_{nk})$, $(n, k = 1, 2, 3, \ldots.)$ as

$$\left.\begin{aligned} a_{nn} &= \frac{1}{n} \\ a_{nk} &= 0 \qquad \text{if } k = n \end{aligned}\right\} \qquad (n, k = 1, 2, 3, \ldots.).$$

Then certainly $A \in (l, l)$. Also

$$l_A = \{x = (x_k) : \sum_{k=1}^{\infty} \frac{|x_k|}{k} \text{ converges}\}.$$

so that l_A is solid. However, $l_A \neq l$

(take $x_k = \dfrac{1}{k}$ for $k = 1, 2, 3, \ldots.$). and $l_A \not\supset w$ (take $x_k = 1$ for $k = 1,2,3,\ldots$) So the first conjecture false.

We thank Professor Brian Kuttner for his interest in this work.

REFERENCES

1 BROWN, H.I.: Entire Methods of Summation, Comp. Math. 21 (1969), 35–42

2 COOKE, R.G.: Infinite matrices and sequence spaces, Dover, 1955

3 WILANSKY, A.: Functional Analysis, Blaisdıll, 1964.

St. Xavier's College
Palayamkottai — 627 002,
India,

and

Nazareth Margoschis College
Nazareth 628 617
India.