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SOME INTEGRAL REPRESENTATIONS OF THE GENERALIZED MULTIDIMENSIONAL WHITTAKER TRANSFORM INVOLVING PRODUCT OF TWO MULTIVARIABLE H-FUNCTIONS

By

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SOME INTEGRAL REPRESENTATIONS OF THE GENERALIZED MULTIDIMENSIONAL WHITTAKER TRANSFORM INVOLVING PRODUCT OF TWO MULTIVATIABLE H-FUNCTIONS

by

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ABSTRACT:

In the present paper, we have introduced a generalized multidimensional Whittaker transform, Laplace transform and Hankel transform, involving the product of two multivariable H-functions as kernel. We have discussed some theorems on multidimensional integral representations for the generalized Whittaker transform. Illustrative examples and corollary have also been included.

INTRODUCTION

We have discussed the multidimensional integral transform

$$\varnothing\ [f{:}p_1,\ldots,p_r] = \int_0^\infty \cdots \int_0^\infty K(p_1,\ldots,p_r;\ x_1,\ldots,x_r)$$

$$f(x_1,\ldots,x_r)\ dx_1\ldots dx_r, \qquad (1.1)$$

where the kernel $K(p_1,\ldots,p_r;\ x_1,\ldots,x_r)$ is the product of two multivariable H-functions.

The multivariable H-function defined by Srivastava and Panda [152] and Prasad and Sing [9] etc has been defined as follows:

$$H\begin{matrix} o,n:\ (V', W');\ldots;\ (V^{(r)}, W^{(r)} \\ p,q:\ [X',Y'];\ldots;\ [X^{(r)}, Y^{(r)}] \end{matrix}\left[x_1,\ldots,x_r \,\middle|\, \begin{matrix} \{(a_p;\ \alpha'_p,\ldots,\alpha_p^{(r)})\}: \\ \{(b_q;\ \beta'_q,\ldots,\beta_q^{(r)})\}: \end{matrix}\right.$$

$$\left.\begin{matrix} \{(A'_{X'}, \eta'_{X'})\} ;\ldots; \{(A^{(r)}_{X^{(r)}}, \eta^{(r)}_{X^{(r)}})\} \\ \\ \{(B'_{Y'}, \xi_{Y'})\} ;\ldots; \{(B^{(r)}_{Y^{(r)}}, \xi^{(r)}_{Y^{(r)}})\} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \varnothing_1(s_1), \ldots, \varnothing_r(s_r)\, \psi(s_1, \ldots, s_r)\, x_1^{s_1}, \ldots, x_r^{s_r}$$

$$dx_1 \ldots dx_r,\ \omega = \sqrt{-1}, \tag{1.2}$$

where

$$\varnothing_i(s_i) = \frac{\prod_{j=1}^{V(i)} \Gamma(B_j^{(i)} - \xi_j^{(i)} s_i) \prod_{y=1}^{W(i)} \Gamma(1 - A_j^{(i)} + \eta_j^{(i)} s_i)}{\prod_{j=V(i)+1}^{Y(i)} \Gamma(1 - B_j^{(i)} + \xi_j^{(i)} s_i) \prod_{j=W(i)+1}^{X(i)} \Gamma(A_j^{(i)} - \eta_j^{(i)} s_i)}, \tag{1.3}$$

$i = 1, \ldots, r,$

$$\psi(s_1, \ldots, s_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} s_i)}{\prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} s_i) \prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} s_i)},$$

where (i) stands for the number of dashes e.g. $a^{(1)} = a'$, $a^{(2)} = a''$ and so an, $\{(A_X, \eta_X)\}$ and $\{(a_p; \alpha'_p, \ldots, \alpha_p^{(\gamma)})\}$ abbreviate to sequence of X and p–parameters. An empty product is interpreted as unity; the coefficients $\alpha_j^{(i)}$, $\beta_j^{(i)}$, $\eta_j^{(i)}$, $\xi_j^{(i)}$ occurring in (1.3) and (1.4) are positive reals and n, p, q, $V^{(i)}$, $W^{(i)}$, $X^{(i)}$, $Y^{(i)}$ are integers such that $0 \leq n \leq p$, $q \geq 0$, $0 \leq V^{(i)} \leq Y^{(i)}$, $0 \leq W^{(i)} \leq X^{(i)}$, $i = 1, \ldots, r$. The contour L_i in the complex s_i –plane, is of Mellin–Barnes type which runs from $-w\infty$ to $+w\infty$ with identations, if necessary in such a manner that all the poles of $\Gamma(B_j^{(i)} - \xi_j^{(i)} s_i)$; $j = 1, \ldots, V^{(i)}$ are to the right and those of $\Gamma(1 - A_j^{(i)} + \eta_j^{(i)} s_i)$, $j = 1, \ldots, W^{(i)}$ and $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} s_i)$, $j = 1, \ldots, n$ to the left of L_i. The points $x_i = 0$, $i = 1, \ldots,$

r being tacitly excluded, the multivariable H–function (1.2) converges absolutely if

$$| \arg x_i | < \tfrac{1}{2} U_i \pi, \; U_i > 0, \; i = 1, \ldots, r, \tag{1.5}$$

where

$$U_i = \sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{V(i)} \xi_j^{(i)} -$$

$$\sum_{j=V(i)+1}^{Y(i)} \xi_j^{(i)} + \sum_{j=1}^{W(i)} \eta_j^{(i)} - \sum_{j=W(i)+1}^{X(i)} \eta_j^{(i)} > 0, \; i = 1, \ldots, r. \tag{1.6}$$

It is easily verified that on the lines of Braaksma [1, p. 246 and p. 279] that

$$H^{o,o:(V',W');\ldots;(V^{(r)},W^{(r)})}_{p,q:[X',Y'];\ldots;[X^{(r)},Y^{(r)}]} \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$$

$$= \begin{cases} 0\left(|x_1|^{\alpha_1} \ldots |x_r|^{\alpha_r}\right), \max \{|x_1|, \ldots, |x_r|\} \longrightarrow 0 \\ 0\left(|x_1|^{-\beta_1} \ldots |x_r|^{-\beta_r}\right), \min \{|x_1|, \ldots, |x_r|\} \longrightarrow \infty \end{cases} \tag{1.7}$$

$$\alpha_i = \min \text{Re} \left(B_j^{(i)} / \xi_j^{(i)}\right), \; j = 1, \ldots, V^{(i)}, \; i = 1, \ldots, r, \tag{1.8}$$

$$\beta^i = \max \text{Re} \left(A_j^{(i)} - 1\right) / \eta_j^{(i)}, \; j = 1, \ldots, W^{(i)}, \; i = 1, \ldots, r. \tag{1.9}$$

The kernel of the transform (1.1) is given by

$$K(p_1, \ldots, p_r; x_1, \ldots, x_r) = \prod_{i=1}^{r} (p_i x_i)^{\eta_i - 1} e^{-\sum_{i=1}^{r} l_i p_i x_i}$$

$$H^{(1)}\left[z_1 (p_1 x_1)^{\sigma_1}, \ldots, z_r (p_r x_r)^{\sigma_r}\right] H^{(2)}\left[c_1 (p_1 x_1)^{\mu_1}, \ldots, c_r (p_r x_r)^{\mu_r}\right], \tag{1.10}$$

where

$$H^{(1)}\left[z_1 (\rho_1 x_1)^{\sigma_1}, \ldots, z_r (p_r x_r)^{\sigma_r}\right] = H^{o,o:(1,N');\ldots;(1,N^{(r)})}_{P,Q:[P',Q'+1];\ldots;[P^{(r)},Q^{(r)}+1]}$$

$$\left[z_1(p_1x_1)^{\sigma_1},\ldots,z_r(p_rx_r)^{\sigma_r} \middle| \begin{array}{l} \{(e_p; E'_p,\ldots,E_p^{(r)})\}: \{(C'_{p'},\gamma'_{p'})\}; \\ \{(f_Q, F'_Q,\ldots,F_Q^{(r)})\}: (D'_o, \delta'^{o}), \{(D'_{Q'},\delta'_{Q'})\}; \end{array} \right.$$

$$\left. \begin{array}{l} \ldots; \{(C^{(r)}_{P^{(r)}}, \gamma^{(r)}_{P^{(r)}})\} \\ \ldots; (D_o^{(r)}, \delta_o^{(r)}), \{(D^{(r)}_{Q^{(r)}}, \delta^{(r)}_{Q^{(r)}})\} \end{array} \right] \tag{1.11}$$

and

$$H^{(2)}[C_1(p_1x_1)^{\mu_1},\ldots,C_r(p_rx_r)^{\mu_r}] = H^{o,o: (V', W');\ldots;(V^{(r)}, W^{(r)})}_{p, q: [X', Y'];\ldots;[X^{(r)}, Y^{(r)}]}$$

$$\left[c_1(p_1x_1)^{\mu_1},\ldots,c_r(p_rx_r)^{\mu_r} \middle| \begin{array}{l} \{(a_p; \alpha'_p,\ldots,\alpha_p^{(r)})\} \quad : \\ \{(b_q; \beta'_q,\ldots,\beta_q^{(r)})\} \quad : \end{array} \right.$$

$$\left. \begin{array}{l} \{(A'_{X'}, \eta'_{X'})\}; \ldots; \{(A^{(r)}_{X^{(r)}}, \eta^{(r)}_{X^{(r)}})\} \\ \{(B'_{Y'}, \xi'_{Y'})\};\ldots;\{(B^{(r)}_{Y^{(r)}}, \xi^{(r)}_{Y^{(r)}})\} \end{array} \right] \tag{1.12}$$

The integral transform (1.1) exists provided that $(\sigma_i) > 0$, $(\mu_i) > 0$, $i = 1, \ldots, r$, $|\arg (c_ip_i)^{\mu_i}| > \frac{1}{2}U_i\pi$, $U_i > 0$, $|\arg (z_ip_i)^{\sigma_i}| < \frac{1}{2} V_i\pi$, $V_i > 0$, $\mathrm{Re}\,(\eta_i + \sigma_i \frac{D_o^{(i)}}{\delta_o^{(i)}} + \mu_i\alpha_i + \nu_i) > 0$, $\mathrm{Re}\,(\eta_i + \mu_i \beta_i + \sigma_i \beta_i^* + \nu_i) < 0$, $(i = 1, \ldots, r)$ where U_i, α_i and β_i are given by equations (1.6), (1.8) and (1.9) respectively and β_i^* and V_i are given by

$$\beta_i^* = \max \mathrm{Re}\,(C_j^{(i)} - 1)/\gamma_j^{(i)}, \; j = 1,\ldots, N^{(i)}; \; i = 1,\ldots, r, \tag{1.13}$$

$$V_i = -\sum_{j=1}^{P} E_j^{(i)} - \sum_{j=1}^{Q} F_j^{(i)} + \delta_o^{(i)} - \sum_{j=1}^{Q^{(i)}} \delta_j^{(i)} + \sum_{j=1}^{N^{(i)}} \gamma_j^{(i)} - \sum_{j=N(i)+1}^{P^{(i)}} \gamma_j^{(i)} > 0, \tag{1.14}$$

$$f(x_1, \ldots, x_r) = 0(|x_1|^{\nu_1} \ldots |x_r|^{\nu_r}), \text{ for small } x_1, \ldots, x_r,$$

$$= 0(|x_1|^{\nu_1} \ldots |x_r|^{\nu_r}), \text{ for large } x_1, \ldots, x_r.$$

The new generalized Whittaker transform defined by us is as follows

$$\varnothing_1 [f: p_1, \ldots, p_r] = W - MT \; [f(x_1, \ldots, x_r)]$$

$$= \int_0^\infty \cdots \int_0^\infty K_1 (p_1, \ldots, p_r; x_1, \ldots, x_r)$$

$$f(x_1, \ldots, x_r)\, dx_1 \ldots dx_r,$$

where the kernel $K_1 (p_1, \ldots, p_r; x_1, \ldots, x_r)$ is taken as

$$\prod_{i=1}^{r} (p_i x_i)^{\eta_i - 1} \exp\left(-\tfrac{1}{2} p_1 x_1 + \sum_{i=2}^{r} p_i x_i\right) W_{k,m} (p_1 x_1)\; H^{(1)} [z_1 (p_1 x_1)^{\sigma_1},$$

$$\ldots, z_r (p_r x_r)^{\sigma_1}]\; H^{(2)} [c_1 (p_1 x_1)^{\mu_1}, \ldots, c_r (p_r x_r)^{\mu_1}].$$

The results arrived at have been given in the form of several theorems; by using different integral representations for $W_{k,m}(px)$ due to Whittaker and Watson [13] and Meijer [5-6] on the right hand side of the integral transform

$$\varnothing_1 [f: p_1, \ldots, p_r] = W\text{-}MT \; [f(x_1, \ldots, x_r)],$$

and interchanging the order of integration under suitable restriction.

We shall also use the series expansion of $H^{(1)} [z_1 (p_1 x_1)^{\sigma_1}, \ldots,$ $z_r (p_r x_r)^{\sigma_r}]$ which can be deduced with the help of the results given by Saxene [11] and Mukerjee and Prasad [7, p.6], as follows:

$$H^{(1)} [z_1 (p_1 x_1)^{\sigma_1}, \ldots, z_r (p_r x_r)^{\sigma_r}] = \frac{1}{\delta_0^{(1)} \ldots \delta_0^{(r)}} \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \varnothing (\rho\nu_1, \ldots, \rho\nu_r)$$

$$\prod_{i=1}^{r} \left\{ \theta_i (\rho\nu_i) \frac{(-1)^{\nu_i}}{\nu_i!} z_i^{\rho\nu_i} (p_i x_i)^{\sigma_i \rho\nu_i} \right\},$$

$$\rho\nu_i = \frac{D_0^{(i)} + \nu_i}{\delta_0^{(i)}} \quad i = 1, \ldots, r, \tag{1.15}$$

where

$$\theta_i(s_i) = \frac{\prod_{j=1}^{N(i)} \Gamma(1-C_j^{(i)} + \gamma_j^{(i)} s_i)}{\prod_{j=1}^{Q(i)} \Gamma(1-D_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=N(i)+1}^{P(i)} \Gamma(C_j^{(i)} - \gamma_j^{(i)} \varepsilon_i)}, \quad i=1,\ldots,5, \quad (1.16)$$

$$\varnothing(s_1,\ldots,s_r) = \left[\prod_{j=1}^{P} \Gamma(e_j - \sum_{j=1}^{r} E_j^{(i)} s_i) \prod_{j=1}^{Q} \Gamma(1-f_j + \sum_{i=1}^{r} F_j^{(i)} s_i) \right]^{-1} \quad (1.17)$$

$\sigma_i > 0$, $|\arg(z_i p_i)^{\sigma_i}| < \frac{1}{2} V_i \pi$, $V_i > 0$ $(= 1, \ldots, r)$, where V_i is given by equation (1.14) and the parameters bear the similar meanings as given in (1.2). We shall also denote the generalized Laplace multiple transform of $f(x_1, \ldots, x_r)$ as

$$L - MT\,[f(x_1,\ldots,x_r)] = \int_0^\infty \cdots \int_0^\infty e^{-\sum_{i=1}^{r} p_i x_i} H^{(i)}[z_1 (p_1 x_1)^{\sigma_1}, \ldots, z_r (p_r x_r)^{\sigma^r}]$$

$$H^{(2)}[c_1 (p_1 x_1)^{\mu_1}, \ldots, c_r (p_r x_r)^{\mu_r}] \, f(x_1, \ldots, c_r) \, dx_1 \ldots dx_r.$$

We shall also denote $\varnothing\,[f: p_1, \ldots, p_r]$ as follows:

$$\Phi \begin{matrix} k,m; \eta_1, \ldots, \eta_r; o,o: (V', W'); \ldots; (V^{(r)}, W^{(r)}); o,o; (1, N'); \ldots; (1, N^{(r)}) \\ c_1, \ldots, c_r; p, q: [X', Y']; \ldots; [X^{(r)}, Y^{(r)}]; z_1, \ldots, z_r; P, Q: \\ (p_1, \ldots, p_r) \\ [P', Q'+1]; \ldots; [P^{(r)}, Q+1] \end{matrix}$$

2. Theorem 1. If

$$\varnothing\,[f: p_1, \ldots, p_r] = W\text{–}MT\,[f(x_1, \ldots, x_r)] \quad (2.1)$$

and

$$g(p_1, \ldots, p_r; y) =$$

$$L\text{–}MT \left[x_1^{\eta_1 - m - \frac{1}{2}} \prod_{i=2}^{r} (x_i)^{\eta_i - 1} (x_1 + y)^{k + m - \frac{1}{2}} f(x_1, \ldots, x_r) \right] \quad (2.2)$$

then

$$\varnothing_1\ [f: p_1, \ldots, p_r] = \frac{(p_1)^{\eta_1 + m - \frac{1}{2}} \prod_{i=2}^{r} (p_i)^{\eta_i - 1}}{\Gamma(\frac{1}{2} - k + m)}$$

$$\int_0^\infty e^{-p_1 y}\, y^{-k + m - \frac{1}{2}}\, g(p_1, \ldots, p_r; y)\, dy, \qquad (2.3)$$

provided that $g(p_1, \ldots, p_r; y)$ defined by (2.2.) exists, $\mathrm{Re}(p_i) > 0$, $(\mu_i) > 0$, $(\sigma_i) > 0$, $i = 1, \ldots, r$; $\mathrm{Re}(-k + m + \frac{1}{2}) > 0$, $\mathrm{Re}(\eta_1 + \mu_1' + \mu_1 \alpha_1 + \sigma_1 \frac{D'_0}{\delta'_0} - m + \frac{1}{2}) > 0$, $\mathrm{Re}(\eta_i + \mu_i' + \mu_i \alpha_i + \sigma_i \frac{D_0^{(i)}}{\delta_0^{(i)}}) > 0$

$(i = 2, \ldots, r)$, $|\arg (c_i p_i)^{\mu_i}| > \frac{1}{2} U_i \pi$, $U_i > 0$, $|\arg (z_i p_i)^{\sigma_i} < \frac{1}{2} V_i \pi$, $V_i > 0$, where α_i, U_i and V_i are given by equations (1.8), (1.6) and (1.14) respectively, where

$$f(x_1, \ldots, x_r) = \begin{cases} \prod_{i=1}^{r} (x_i)^{\mu'_i}, & \mathrm{Re}(\mu'_i) > 0, \text{ for small values of } x_1, \ldots, x_r; \\ \left(e^{-\sum_{i=1}^{r} \mu''_i x_i} \right), & \mathrm{Re}(\mu_i'') > 0, \text{ for large values of } x_1, \ldots, x_r; \end{cases}$$

$f(x_1, \ldots, x_r)$ is continuous for $x_1, \ldots, x_r > 0$ and the resulting integral in (2.3) is absolutely convergent.

Proof. We have

$$\varnothing_1\ [f: p_1, \ldots, p_r] = \int_0^\infty \cdots \int_0^\infty (p_i x_i)^{\eta_i - 1} \exp\left(-\left(\sum_{i=2}^{r} p_i x_i + \tfrac{1}{2} p_1 x_1\right)\right)$$

$$W_{k,m}(p_1 x_1)\, H^{(1)}\, [z_1 (p_1 x_1)^{\sigma_1}, \ldots, z_r (p_r x_r)^{\sigma_r}]$$

$$H^{(2)}\, [c_1 (p_1 x_1)^{\mu_1}, \ldots, c_r (p_r x_r)^{\mu_r}]\, f(x_1, \ldots, x_r)\, dx_1 \ldots dx_r. \qquad (2.4)$$

Using the integral representation for $W_{k,m}(p_1x_1)$ due to Whittaker and Watson [12, p. 340], i.e.

$$W_{k,m}(p_1x_1) = \frac{e^{-\frac{1}{2}p_1x_1}(p_1x_1)^k}{\Gamma(\frac{1}{2}-k+m)} \int_0^\infty t^{-k+m-\frac{1}{2}} \left(1+\frac{t}{p_1x_1}\right)^{k+m-\frac{1}{2}} e^{-t}dt, \tag{2.5}$$

where Re $(k-\frac{1}{2}-m) \leq 0$ and is not an integer, we have

$$\varnothing_1 [f: p_1, \ldots, p_r] = \int_0^\infty \cdots \int_0^\infty \frac{e^{-p_1x_1}(p_1x_1)^{k+\eta_1-1}}{\Gamma(\frac{1}{2}-k+m)} \prod_{i=2}^{r} (p_ix_i)^{\eta_i-1} e^{\sum_{i=1}^{r} p_ix_i}$$

$$H^{(1)}[z_1(p_1x_1)^{\sigma_1}, \ldots, z_r(p_rx_r)^{\sigma_r}]\, H^{(2)}[c_1(p_1x_1)^{\mu_1}, \ldots, c_r(p_rx_r)^{\mu_r}]$$

$$\left\{\int_0^\infty t^{-k+m-\frac{1}{2}}\left(1+\frac{t}{p_1x_1}\right)^{k+m-\frac{1}{2}} e^{-t}dt\right\} dx_1 \ldots dx_r. \tag{2.6}$$

Substituting $t = p_1y$ in the integral and changing the order of integration, we have

$$\varnothing_1 [f: p_1, \ldots, p_r] = \frac{(p_1)^{\eta_1+m-\frac{1}{2}} \prod_{i=2}^{r} (p_i)^{\eta_i-1}}{\Gamma(\frac{1}{2}-k+m)} \int_0^\infty e^{-p_1y}\, y^{-k+m-\frac{1}{2}}$$

$$\left\{\int_0^\infty \cdots \int_0^\infty e^{-p_1x_1}\, x_1^{\eta_1-m-\frac{1}{2}} \prod_{i=2}^{r} (x_i)^{\eta_i-1} e^{\sum_{i=2}^{r} p_ix_i} (x_1+y)^{k+m-\frac{1}{2}}\right.$$

$$H^{(1)}[z_1(p_1x_1)^{\sigma_1}, \ldots, z_r(p_rx_r)^{\sigma_r}]\, H^{(2)}[c_1(p_1x_1)^{\mu_1}, \ldots, c_r(p_rx_r)^{\mu_r}]$$

$$\left. f(x_1, \ldots, x_r)\, dx_1 \ldots dx_r \right\} dy$$

$$= \frac{(p_1)^{\eta_1+m-\frac{1}{2}} \prod_{i=2}^{r} (p_i)^{\eta_i-1}}{\Gamma(\frac{1}{2}-k+m)} \int_0^\infty e^{-p_1 y} y^{-k+m-\frac{1}{2}} g(p_1, \ldots, p_r; y)\, dy, \tag{2.7}$$

provided that $g(p_1, \ldots, p_r, y)$ defined by (2.2) exists.

Corollary 1. On taking $r = 1$, $P = Q = 0 = P' = Q' = D'_0$, $\sigma_1 = 1 = \delta'_0$, $z_1 \to 0$ in $H^{(1)}[z_1 (p_1 x_1)^{\sigma_1}, \ldots, z_r (p_r x_r)^{\sigma_r}]$ in Theorem 1, we arrive at the theorem recently studied by Maurya [4, p. 175].

Corollary 2. On taking single integral in place of multiple integral $\eta_1 = \eta$, $\eta_2 = \ldots = \eta_r = 1$, $p_1 = \ldots = p_r = p$, $x_1 = \ldots = x_r = x$ in Corollary 1 of the Theorem 1, we get the theorem recently studied by Prasad and Singh [10, p. 295].

Example. Let.

$$f(x_1, \ldots, x_r) = \prod_{i=1}^{r} x^{\rho_i},$$

then

$$\varnothing_1 [f: p_1, \ldots, p_r] = \frac{\prod_{i=1}^{r} (p_i)^{\eta_i + \sigma_i \rho \nu_i - 1}}{\delta'_0 \ldots \delta_0^{(r)}} \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \varnothing(\rho\nu_1, \ldots, \rho\nu_r)$$

$$\prod_{i=1}^{r} \left\{ \theta_i(\rho\nu_i) \frac{(-1)^{\nu_i} z_i^{\rho\nu_i}}{\nu_i!} \right\}$$

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} (x_i)^{\eta_i + \rho_i + \sigma_i \rho \upsilon_i - 1} e^{-(\frac{1}{2} p_1 x_1 + \sum_{i=2}^{r} p_i x_i)} W_{k,m}(p_1 x_1)$$

$$H^{(2)} [c_1 (p_1 x_1)^{\mu_1}, \ldots, c_r (p_r x_r)^{\mu_r}]\, dx_1 \ldots dx_r. \tag{2.8}$$

To evaluate the right hand side of (2.8), we use the result

$$\int_0^\infty e^{-\frac{1}{2}px}\, W_{k,m}(px)\, x^{n-1}\, dx = \Gamma\left(\tfrac{1}{2}+n+m\right) / p^n\, \Gamma(n-k+1) \quad (2.9)$$

for first integral and the definition of Gamma function, for the other integrals, we obtain

$$\varnothing_1\, [f: p_1, \ldots, p_r] = \frac{1}{\delta'_o \ldots \delta_o^{(r)}} \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \varnothing\, (\rho\nu_1, \ldots, \rho\nu_r)$$

$$\prod_{i=1}^{r} \left\{ \theta_i\, (\rho\nu_i)\, \frac{(-1)^{\nu_i}\, z_i^{\rho\nu_i}\, p_i^{-\rho_i - 1}}{\nu_i!} \right\} H^{o,\, o:\, (V',\, W'+2);\, (V'',\, W''+1);}_{p,q:\ [X'+2, Y'];\, [X''+1, Y''];\, \ldots}$$

$$\left.\begin{matrix} (V^{(r)}, W^{(r)}+1) \\ [X^{(r)}+1, Y^{(r)}] \end{matrix}\right. \left[c_1, \ldots, c_r \,\middle|\, \begin{matrix} \{(a_p; \alpha'_p, \ldots, \alpha_p^{(r)})\}: \\ \{(b_q; \beta'_q, \ldots, \beta_q^{(r)})\}: \end{matrix} \right.$$

$$(\tfrac{1}{2}-\rho_1-\eta_1-\sigma_1\rho\nu_1+m, \mu_1),\ \{(A'_{X'}, \eta'_{X'})\};\ (1-\rho_2-\sigma_2\rho\nu_2-\eta_2, \mu_2),$$

$$(-\rho_1-\eta_1-\sigma_1\rho\nu_1+k, \mu_1),\ \{(B'_{Y'}, \xi'_{Y'})\};\ \{(B''_{Y''}, \xi''_{Y'},)\};\ \ldots;$$

$$\{(A''_{X''}, \eta''_{X''})\};\ \ldots;$$

$$\left. \begin{matrix} (1-\eta_r-\sigma_r\rho\nu_r-\rho_r, \mu_r), & \{(A^{(r)}_{X^{(r)}}\ \eta^{(r)}_{X^{(r)}})\} \\ & \{(B^{(r)}_{Y^{(r)}},\ \xi^{(r)}_{Y^{(r)}})\} \end{matrix} \right], \quad (2.10)$$

provided that $(\mu_i) > 0$, $(\sigma_i) > 0$, $i = 1, \ldots, r$, $\text{Re}\,(\rho_1 + \eta_1 + \sigma_1\rho\nu_1 + \mu_1\alpha_1 + \frac{1}{2} + m) > 0$ $\text{Re}\,(\rho_i + \eta_i + \sigma_i\rho\nu_i + \mu_i\alpha_i) > 0$ $(i = 2, \ldots, r)$, $|\arg\, (C_i p_i)^{\mu_i}| < \frac{1}{2} U_i\pi$, $U_i > 0$, $|\arg\, (z_i p_i)^{\sigma_i}| < \frac{1}{2} V_i\pi$, $V_i > 0$, $i=1, \ldots, r$, where α_i, U_i and V_i are given by the equations (1.8), (1.6) and (1.14) respectively.

Now

$$g\, (p_1, p_1, \ldots, p_r; y) = \frac{1}{\delta'_o \ldots \delta_o^{(r)}} \sum_{\nu_1, \ldots, \nu^r = 0}^{\infty} \varnothing\, (\rho\nu_1, \ldots, \rho\nu_r)$$

$$\prod_{i=1}^{r}\left\{\theta_i(\rho\nu_i)\frac{(-1)^{\nu_i} z_i^{\rho\nu_i}}{\nu_i!}\right\}\int_0^\infty\cdots\int_0^\infty x_1^{\eta_1+\sigma_1\rho\nu_1+\rho_1-m-\frac{1}{2}}$$

$$\prod_{i=2}^{r}(x_i)^{\eta_i+\sigma_i\rho\nu_i+\rho_i-1} e^{-\sum_{i=1}^{r} p_i x_i} H^{(2)}\left[c_1(p_1x_1)^{\mu_1},\ldots,c_r(p_rx_r)^{\mu_r}\right]$$

$$(x_1+y)^{k+m-\frac{1}{2}} dx_1\ldots dx_r.$$

$$=\frac{1}{(2\pi\omega)^r\,\delta'_0\ldots\delta_0^{(r)}}\sum_{\nu_1,\ldots,\nu_r=0}^{\infty}\varnothing(\rho\nu_1,\ldots,\rho\nu_r)\prod_{i=1}^{r}\left\{\theta_i(\rho\nu_i)\right.$$

$$\left.\frac{(-1)^{\nu_i} z_i^{\rho\upsilon_i}}{\nu_i!}\right\}\int_{L_1}\cdots\int_{L_r}\varnothing_1(s_1)\ldots\varnothing_r(s_r)\,\psi(s_1,\ldots,s_r)$$

$$(c_1p_1)^{\mu_1 s_1}\ldots(c_rp_r)^{\mu_r s_r}\left\{\int_0^\infty e^{-\sum_{i=1}^{r} p_i x_i}(x_1+y)^{k+m-\frac{1}{2}}\right.$$

$$x_1^{\eta_1+\sigma_1\rho\nu_1+\rho_1+\mu_1 s_1-\frac{1}{2}}\prod_{i=2}^{r}(x_i)^{\eta_i+\sigma_i\rho\nu_i+\rho_i+\mu_i s_i-1}$$

$$\left. dx_1\ldots dx_r\right\} ds_1\ldots ds_r. \tag{2.11}$$

Now, evaluating the inner integral with the help of Erdelyi [3, vol. I, p. 139, Eq. (22)], viz.

$$\int_0^\infty e^{-pt}(t+a)^{2\mu-1} t^{2\nu-1} dt=\Gamma(2\upsilon)\, a^{\mu+\nu-1} p^{-\mu-\nu-\frac{1}{2}} e^{ap}$$

$$W_{\mu-\nu,\,\mu+\nu-\frac{1}{2}}(ap), \tag{2.12}$$

provided that $Re(p) > 0$; $t > 0$, $Re(\nu) > 0$; $|\arg a| < \pi$; (2.10) reduces to

$$g(p_1, \ldots, p_r; y) = \frac{1}{\prod_{i=2}^{r} (p_i)^{\eta_i + \sigma_i \rho \nu_i + \rho_i} (2\pi\omega)^{\gamma} \delta_0^{r} \ldots \delta_0^{(r)}}$$

$$\sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \varnothing(\rho\nu_1, \ldots, \rho\nu_r) \prod_{i=1}^{r} \left\{ \theta_i(\rho\nu_i) \frac{(-1)^{\nu_i} z_i^{\rho\nu_i}}{\nu_i!} \right\} \int_{L_1} \ldots \int_{L_r}$$

$$\varnothing_1(s_1) \ldots \varnothing_r(s_r) \psi(s_1, \ldots, s_r)(c_1 p_1)^{\mu_1 s_1} \ldots (c_r p_r)^{\mu_r s_r} \prod_{i=2}^{r}$$

$$\Gamma(\eta_i + \sigma_i\rho\nu_i + \rho_i + \mu_i s_i) \Gamma(2\nu'_1)(y)^{\mu'_1 + \nu'_1 - 1} (p_1)^{-\mu'_1 - \nu_i} e^{-\frac{1}{2} p_1 y}$$

$$W_{\mu'_1 - \nu_i, \mu'_1 + \nu_i - \frac{1}{2}} (p,y)\, ds_1 \ldots ds_r, \tag{2.13}$$

where $\mu_1' = \frac{1}{2}(k + m + \frac{1}{2})$, $\nu'_1 = \frac{1}{2}(\eta_1 + \sigma_1\rho\nu_1 + \rho_1 + \mu_1 s_1 - m + \frac{1}{2})$; Re $(p_i) > 0$,

Re $(\frac{1}{4} + \frac{1}{2}\rho_1 + \frac{1}{2}\eta_1 + \frac{1}{2}\sigma_1\rho\nu_1 - \frac{1}{2}m) > 0$ and $|\arg y| > \pi$.

Now, using the theorem, the right hand side of (2.3) is

$$\int_0^{\infty} e^{-p_1 y} y^{-k + m - \frac{1}{2}} g(p_1, \ldots, p_r; y)\, dy$$

$$= \frac{1}{(2\pi\omega)^r \delta_0' \ldots \delta_0^{(r)}} \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \varnothing(\rho\nu_1, \ldots, \rho\nu_r) \prod_{i=1}^{r} \left\{ \theta_i(\rho\nu_i) \right.$$

$$\left. \frac{(-1)^{\nu_i} z_i^{\rho\nu_i} p_i^{-\rho_i - 1}}{\nu_i!} \right\} \int_{L_1} \ldots \int_{L_r} \varnothing_1(s_1) \ldots \varnothing_r(s_r) \psi(s_1, \ldots, s_r)$$

$$(c_1 p_1)^{\mu_1 s_1} \ldots (c_r p_r)^{\mu_\gamma s_r} \Gamma(2\nu'_i) \prod_{i=2}^{r} \Gamma(\eta_i + \rho_i + \sigma_i\rho\nu_i + \mu_i s_i)$$

$$\left\{ \int_0^{\infty} e^{\frac{1}{2} p_1 y} y^{\mu'_1 + \nu'_1 - 1} W_{\mu'_1 - \nu'_1, \mu'_1 + \nu'_1 - 1} (p_1 y)\, dy \right\}$$

$$ds_1 \ldots ds_r. \tag{2.14}$$

Evaluating the inner integral in (2.14) with the help of the known integral (2.9) and interpreting the result with the definition of multivariable H-function, we arrive at the right hand side of (2.10). Thus the theorem is verified.

3. Theorem 2. If

$$\varnothing_1 [f: p_1, \ldots, p_r] = \text{W–MT} \; [f(x_1, \ldots, x_r)], \tag{3.1}$$

and

$$g(p_1, \ldots, p_r; t) = \text{L - MT} \left[x_1^{\eta_1 - \frac{1}{2}} \prod_{i=2}^{r} (x_i)^{\eta_i - 1} K_{2m}(2\sqrt{p_1 x_1}\, t)\, f(x_1, \ldots, x_r) \right], \tag{3.2}$$

then

$$\varnothing_1 [f: p_1, \ldots, p_r] = \frac{4\,(p_1)^{\eta_1 - \frac{1}{2}} \prod_{i=2}^{r} (p_i)^{\eta_i - 1}}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty e^{-t^2}\, t^{-2k}\, g(p_1, \ldots, p_r; t)\, dt, \tag{3.3}$$

provided that $\text{Re}\,(\frac{1}{2} - k + m) > 0$, $\text{Re}\,(\eta_1 + \mu_1 \alpha_1 + \sigma_1 \frac{D'_o}{\delta'_o} + \mu_i + \frac{1}{2} \pm m) > 0$, $\text{Re}\,(\eta_i + \mu_i \alpha_i + \sigma_i \frac{D_o({}_i)}{\delta_o(\gamma)} + \mu'_i) > 0$ $(i = 2, \ldots, r)$,

$|\arg (c_i p_i)^{\mu_i}| < \frac{1}{2} U_i \pi$, $U_i > 0$, $|\arg (z_i p_i)^{\sigma_i}| < \frac{1}{2} V_i \pi$, $V_i > 0$, $i = 1, \ldots, r$, where α_i, U_i and V_i are given by equations (1.8), (1.6) and (1.14) respectively, where

$$f(x_1, \ldots, x_r) = \begin{cases} 0\left(\prod_{i=1}^{r} x_i \right)^{\mu'_i}, \; \text{Re}\,(\mu'_1) > 0, \text{ for small values of } x_1, \ldots, x_r, \\ 0\left(e^{-\sum_{\gamma=1}^{r} \mu''_i \cdot i} \right), \; \text{Re}\,(\mu''_1) > 0, \text{ for large values of } x_1, \ldots, x_r, \end{cases}$$

and the interval in (3.3) is absolutely convergent.

Proof. On using the integral representation for $W_{k,m}(p_1x_1)$ due to Meijer [6, p. 299], viz

$$W_{k,m}(p_1x_1) = \frac{4\,(p_1x_1)^{\frac{1}{2}}\, e^{-\frac{1}{2}p_1x_1}}{\Gamma(\frac{1}{2}-k\pm m)} \int_0^\infty e^{-t^2}\, t^{-2k}\, K_{2m}(2\sqrt{p_1x_1}\,t)\,dt \tag{3.4}$$

where $\mathrm{Re}\,(p_i) > 0$, $\mathrm{Re}\,(\frac{1}{2}-k+m) > 0$; we have

$$\varnothing_1\,[f: p_1,\ldots,p_r] = \frac{4\,(p_1)^{\eta_1-\frac{1}{2}} \prod_{i=2}^{r} (p_i)^{\eta_i-1}}{\Gamma(\frac{1}{2}-k\pm m)} \int_0^\infty \cdots \int_0^\infty$$

$$x_1^{\eta_1-\frac{1}{2}} \prod_{i=2}^{r} (x_i)^{\eta_i-1}\, e^{-\sum_{i=1}^{r}(p_ix_i)}\, H^{(1)}\left[z_1\,(p_1x_1)^{\sigma_1},\ldots, z_r\,(p_rx_r)^{\sigma_r}\right]$$

$$H^{(2)}\left[c_1\,(p_1x_1)^{\mu_1},\ldots, c_r\,(p_rx_r)^{\mu_r}\right] f(x_1,\ldots,x_r) \left\{ \int_0^\infty e^{-t^2}\, t^{-2k} \right.$$

$$\left. K_{2m}(2\sqrt{p_1x_1}\,t)\,dt \right\} dx_1 \ldots dx_r. \tag{3.5}$$

On changing the order of integration in (3.5), we get

$$\varnothing_i\,[f: p_1,\ldots,p_r] = \frac{4\,(p_1)^{\eta_1-\frac{1}{2}} \prod_{i=2}^{r} (p_i)^{\eta_i-1}}{\Gamma(\frac{1}{2}-k\pm m)} \int_0^\infty e^{-t^2}\, t^{-2k}$$

$$\left\{ \int_0^\infty \cdots \int_0^\infty (x_1)^{\eta_i-\frac{1}{2}} \prod_{i=2}^{r} (x_i)^{\eta_i-1}\, e^{-\sum_{i=1}^{r} p_ix_i}\, K_{2m}(2\sqrt{p_1x_1}\,t) \right.$$

$$H^{(1)}\left[z_1\,(p_1x_1)^{\sigma_1},\ldots, z_r\,(p_rx_r)^{\sigma_r}\right] H^{(2)}\left[c_1\,(p_1x_1)^{\mu_1},\ldots, c_r\,(p_rx_r)^{\mu_r}\right],$$

$f(x_1, \ldots, x_r)\, dx_1 \ldots dx_r \}\, dt.$

$$= \frac{4\,(p_1)^{\eta_1 - \frac{1}{2}} \prod_{i=2}^{r} (p_i)^{\eta_i - 1}}{\Gamma(\frac{1}{2} - k \pm m)} \int_0^\infty e^{-t^2}\, t^{-2k}\, g(p_1, \ldots, p_r; t)\, dt. \qquad (3.6)$$

The inversion of the order of integration is justifiable as follows:

The x_1- integrals will be absolutely convergent if $\mathrm{Re}(p_i) > 0$, and $\mathrm{Re}\,(\eta_1 + \mu_1\alpha_1 + \sigma_1 \frac{D'_0}{\delta'_0} + \mu'_1 + \frac{1}{2} \pm m) > 0$, $\mathrm{Re}\,(\eta_i + \mu_i\alpha_i + \sigma_i \frac{D_0^{(i)}}{\delta_0^{(i)}} + \mu_i) > 0$ $(i = 2, \ldots, r)$, where

$$f(x_1, \ldots, x_r) = \begin{cases} 0\left(\prod_{i=1}^{r} x_i^{\mu'_i}\right), \mathrm{Re}(\mu'_i) > 0, \text{ for small values of } x_1, \ldots, x_r, \\ 0\left(e^{\sum_{i=1}^{r} \mu''_i x_i}\right), \mathrm{Re}(\mu''_i) > 0, \text{ for large values of } x_1, \ldots, x_r \end{cases}$$

The t-integral is absolutely convergent, if $\mathrm{Re}\,(\frac{1}{2} - k \pm m) > 0$. Also the resulting integral in (3.3) is given to be absolutely convergent. Hence the inversion of order of integration is justified by de La Valle Poussin's theorem [2, p. 504].

Corollary 1. On taking $r = 1$, $P = Q = 0 = P' = Q' = D_0'$, $\delta'_0 = 1 = \sigma_1$ and $z_1 \to 0$ in $H^{(1)}\,[z_1\,(p_1x_1)^{\sigma_1}, \ldots, z_r\,(p_rx_r)^{\sigma_r}]$ in the Theorem 2, we arrive at the theorem recently studied by Maurya [5, p. 190].

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