COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1: Mathématiques

TOME : 33 ANNÉE : 1984

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by

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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences
de l'Université d'Ankara

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A Theorem For Entire Functions Of Irregular Logarithmic Growth

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(Received January 5, 1984, accepted January 30, 1984)

ABSTRACT

For a non-constant entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z = r\, e^{i\theta}$, the logarithmic order ρ^* $(1 \leq \rho^* \leq \infty)$ and lower logaritghmic order λ^* $(1 \leq \lambda^* \leq \infty)$ are given by the limit superior and limit inferior of $\{\log \log M(r)\} / (\log \log r)$, as $r \to \infty$, respectively, where $M(r) = \max_{|z|=r} |f(z)|$. In this paper we derive a formula for λ^*—type t_{λ^*} $\{= \liminf_{r \to \infty} (\log M(r) / (\log r)^{\lambda^*(r)})\}$, $0 < t_{\lambda^*} < \infty$, $1 < \lambda^* < \infty$, in terms of the coefficients a'_ns Here $\lambda^*(r)$ is the lower logarithmic proximate order of $f(z)$ and satisfying the conditions: (i) $\lambda^*(r) \to \lambda^*$, as $r \to \infty$, (ii) $r(\log r)(\lambda^*(r))' \log \log r \to 0$, as $r \to \infty$.

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $z = r\, e^{i\theta}$, be a non-constant entire function of order zero. As usual we put $M(r) = \max_{|z|=r} |f(z)|$ and $\mu(r) = \max_{n \geq 0} \{|a_n| r^n\}$. The logarithmic order ρ^* and lower logarithmic order λ^* are defined as [2]:

$$(1.1) \quad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(r)}{\log \log r} = \begin{matrix} \rho^* \\ \lambda^* \end{matrix}, \quad 1 \leq \lambda^* \leq \rho^* \leq \infty.$$

For functions of logarithmic order ρ^*, $1 < \rho^* < \infty$, Rahman [1] defined the logarithmic type T^* and lower logarithmic type t^* as:

$$(1.2)\quad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{(\log r)^{\rho^*}} = \begin{matrix}T^*\\ t^*\end{matrix},\ 0 \le t^* \le T^* \le \infty.$$

It is easily seen that if f(z) is of irregular logarithmic growth, that is, $1 < \lambda^* < \rho^* < \infty$, then $t^* = 0$. If this is the case, we define λ^*—type of f(z) with respect to $\lambda^*(r)$ as:

$$(1.3)\quad \lim_{r\to\infty} \inf \frac{\log M(r)}{(\log r)^{\lambda^*(r)}} = t_\lambda^*,\ 0 < t_\lambda^* < \infty.$$

The lower logarithmic proximate order $\lambda^*(r)$ of f(z) satisfying the following two conditions:

$$(1.4)\quad \lim_{r\to\infty} \lambda^*(r) = \lambda^*,$$

$$(1.5)\quad \lim_{r\to\infty} r(\log r)\ (\lambda^*(r))'\ \log\log r = 0.$$

In this note, we shall find a coefficient formula for t_{λ^*} in terms of the coefficients a_n's.

2. To state our result in precise terms, we introduce a function $\varnothing(t)$, which is defined as the unique solution (for $t > t_0$) of the equation
(2.1) $t = (\log r)^{\lambda^*(r)-1}$.

Now, we prove the following:

THEOREM. Let f(z) be an entire function of lower logarithmic order $\lambda^*(1 < \lambda^* < \infty)$. *Then a necessary and sufficient condition that* t^*_λ *be the* λ^**—type of* f(z) *with respect to* $\lambda^*(r)$ *is that*

$$(2.2)\quad \lim_{n\to\infty} \inf \frac{n \log \varnothing(n)}{\log |a_n|^{-1}} = \left\{\frac{\lambda^*}{\lambda^*-1}\right\} (\lambda^*\ t_\lambda^*)^{1/(\lambda^*-1)}$$

provided $|a_n/a_{n+1}|$ *forms a non-decreasing function of* n *for* $n > n_0$.

PROOF. First we show that if $h > 0$, then

$$(2.3)\quad \lim_{t\to\infty} \frac{\log \varnothing(h\,t)}{\log \varnothing(t)} = h^{1/(\lambda^*-1)}.$$

From (2.1), we have

$$\log t = \{\lambda^*(r) - 1\} \log\log r.$$

Therefore,

$$\frac{d(\log t)}{d(\log\log r)} = \lambda^*(r) + r(\log r)(\lambda^*(r))' \log\log r - 1.$$

On using (1.4) and (1.5), we get

$$\lim_{r\to\infty} \frac{d(\log t)}{d(\log\log r)} = \lambda^* - 1$$

and so,

$$\lim_{t\to\infty} \frac{d(\log\log \varnothing(t))}{d(\log t)} = \frac{1}{\lambda^*-1}.$$

Thus, for any $\epsilon > 0$ and $t > t_0$.

$$\left\{\frac{1}{\lambda^*-1} - \epsilon\right\} d(\log t) < d(\log\log \varnothing(t)) < \left\{\frac{1}{\lambda^*-1} + \epsilon\right\} d(\log t),$$

which on integration from t to ht gives,

$$\left\{\frac{1}{\lambda^*-1} - \epsilon\right\} \log h < \log\left\{\frac{\log \varnothing(ht)}{\log \varnothing(t)}\right\} < \left\{\frac{1}{\lambda^*-1} + \epsilon\right\} \log h$$

Hence, $\lim_{t\to\infty} \frac{\log \varnothing(ht)}{\log \varnothing(t)} = h^{1/(\lambda^*-1)}.$

It is well known [3] that for functions of finite order,

(2.4) $\log M(r) \approx \log \mu(r)$, as $r \to \infty$.

Hence, $M(r)$ can be replaced by $\mu(r)$ in (1.1), (1.2) and (1.3) etc. Now, from (1.3), we have for $t_1^* < t_\lambda^*$ and for all large values of r,

$$\log \mu(r) > t_1^* (\log r)^{\lambda^*(r)}.$$

In particular, if we take $R_n \leq r < R_{n+1}$, where $R_n = |a_n/a_{n+1}|$, then

$$\log |a_n| > t_1^* (\log r)^{\lambda^*(r)} - n \log r$$

for all large values of n. Let $n = \lambda^* t_1^* (\log r)^{\lambda^*(r)-1}$.

Then, for all $n > n_0$, we have

$$\frac{n \log \varnothing(n)}{\log |a_n|^{-1}} > \left\{\frac{\lambda^*}{\lambda^*-1}\right\} \frac{\log \varnothing(n)}{\log \varnothing\left(\frac{n}{\lambda^* t_1^*}\right)}.$$

Using (2.3) and the fact that $t_\lambda^* - t_1^*$ is arbitrary, we get

$$(2.5) \quad \lim_{n\to\infty} \inf \frac{n \log \varnothing(n)}{\log |a_n|^{-1}} \geq \left\{ \frac{\lambda^*}{\lambda^*-1} \right\} (\lambda^* t_\lambda^*)^{1/(\lambda^*-1)}.$$

We now show that the inequality in (2.5) can not occur. For in that case, a number t_2^* $(t_2^* > t_\lambda^*)$ can be found such that

$$\lim_{n\to\infty} \inf \frac{n \log \varnothing(n)}{\log |a_n|^{-1}} = \left\{ \frac{\lambda^*}{\lambda^*-1} \right\} (\lambda^* t_2^*)^{1/(\lambda^*-1)}.$$

Choosing any number t_3^* between t_2^* and t_λ^* $(t_2^* > t_3^* > t_\lambda^*)$, we have, for all $n > n_0$.

$$\log |a_n| > \left\{ \frac{1-\lambda^*}{\lambda^*} \right\} n \log \varnothing \left(\frac{n}{\lambda^* t_3^*} \right)$$

which with Cauchy's inequality, $M(r), \geq |a_n| r^n$, gives

$$\log M(r) > \left\{ \frac{1-\lambda^*}{\lambda^*} \right\} n \log \varnothing \left(\frac{n}{\lambda^* t_3^*} \right) + n \log r.$$

Taking $n = \lambda^* t_3^* (\log r)^{\lambda^*(r)-1}$, it follows that

$$\lim_{r\to\infty} \inf \frac{\log M(r)}{(\log r)^{\lambda^*(r)}} \geq t_3^*,$$

or, $t_\lambda^* \geq t_3^*$

which contradicts that $t_\lambda^* < t_3^*$. This completes the proof of Theorem.

REMARK. Taking $\lambda^*(r) = \lambda^*$ in the Theorem, we get the following interesting result.

$$\lim_{n\to\infty} \inf \frac{n^{\lambda^*/(\lambda^*-1)}}{\log |a_n|^{-1}} = \left\{ \frac{\lambda^*}{\lambda^*-1} \right\} (\lambda^* t_\lambda^*)^{1/(\lambda^*-1)},$$

where t_λ^* is the λ^*— type of $f(z)$ given by

$$\lim_{r\to\infty} \inf \frac{\log M(r)}{(\log r)^{\lambda^*}} = t_\lambda^*, \quad 0 < t_\lambda^* < \infty.$$

In the end, I wish to thank Dr. G.S. Srivastava for his helpful remarks.

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