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A Criterion For Restricted Ideal Sheaves and on Regular Covering Spaces

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SUMMARY

In this paper a Criterion for restricted ideal sheaves and a necessary and sufficient contidion for regular subcovering spaces are given.

1. A Criterion

In a recent paper [4] we have studied the algebraic structure of restricted analytic sheaves and subsheaves.

Accordingly, the results of the papers [1, 2, 3] may be collected in the following

Theorem 1.1. Let X be a connected complex analytic manifold with fundamental group $F \neq 1$. Let D denote any normal subgroup of F such that F / D is commutative. Then [F,F] is the smallest normal subgroup of theat type. Each D determines a regular covering space of X with fundamental group D and which is a restricted ideal subsheaf of the restricted analytic sheaf A of germs of the totality A (X) of holomorphic functions on X. In particular, [F,F] determines A which is maximal.

The proof of this theorem is substantially contained in the quoted papers and especially in the fundamental theorem [3].

On the other hand [F, F] $\subset$ D is clear. For otherwise there is [a, b] $\notin$ D, a, b, ε F. The elements of F / D being the cosets of D, let A, B, A^{-1}, B^{-1} be those containing a, b, a^{-1}, b^{-1} respectively. But then [A, B] is a coset containing [a, b] and is different from D, and so A, B

do not commute, contrary to the hypothesis. Finally, F / [F, F] is commutative. For, if A, B are any two cosets of [F, F] then [A, B] is a coset containing a commutator [a, b], a ∈ A, b ∈ B and so [A, B] ≡ [F,F]. Hence A, B commute.

Summarizing, we obtain the following

Criterion .F / D is commutative (or a regular covering space of X determined by D is r-ideal subsheaf of A) if and only if D contains [F, F].

We now give examples of normal subgroups of F that do not contain [F, F]. Indeed, if N is any normal subgroup of F, so is [N, N]. In fact, if $n = [a, b] \in [N, N]$, then for every $g \in F$,

$$g^{-1}ng = (g^{-1}ag)(g^{-1}bg)(g^{-1}ag)^{-1}(g^{-1}bg)^{-1} \in [N,N].$$

But clearly, [N,N] does not contain [F,F]. Hence not every regular covering space of X is a r–ideal subsheaf. Note also that [N,N] is smaller than [F,F] but F / [N,N] is not commutative,.

2. *Regular Covering Spaces.* For our purpose we may adopt the following [5]. The criterion is self satisfied as soon as F is a Abelian For then every subgroup D is normal, F/D Abelian, [F, F] = 1.

Definition 2.1. Let Y,X be two connected Hausdorff spaces, π: Y → X is a covering space of X if evergy x ∈ X has an open neighborhoodu such that $\pi^{-1}(U)$ is a disjoint union of open sets s_i in Y, each of which is mapped homoemorphically onto U by π. Such U are said to be evenly covered, and the s_i are called sheets over U.

As immediate consequences:

1. The fibre $\pi^{-1}(x)$ over any point x ε X is discrete.

2. π is local homeomorphic.

3. π is continuous.

π is called a projection. The covering space is usually denoted by (Y,π) or simply by Y if no confusion arizes.

An important property is that open sets are mapped on open sets. In particular, π (Y) is a region (connected open set) on X and (Y, π) is a covering space of π (Y).

Definition 2.2.. Let (Y,π) be a covering space of X, $Y^* \subset Y$ *open* and $\pi^* = \pi\,|Y^* : Y^* \to X$ with $\pi^{*-1}(x) \subset \pi^{-1}(x)$ over any point $x \in X$. Then (Y^*, π^*) is called a subcovering space of Y.

We say that (Y, π) is a stronger covering than (Y^*, π^*) or (Y^*, π^*) is a weaker covering than (Y, π).

Theorem 2.1. Each subcovering of a covering (Y, π) is a covering space of X.

Proof. We need only to show that $\pi^* : Y^* \to X$ is locally topological. Indeed, for every element $\sigma \in Y^*$ there are open neighborhoods V (σ) $\subset$ Y and U $(\pi(\sigma)\,) \subset$ X such that $\pi\,|V: V \to U$ is topological. But then $V^* = V \cap Y^*$ is an open neighborhood of σ in Y^*, $U^* = \pi\,(V^*)$ is an open neighborhood of $\pi\,(\sigma)$ in X and $\pi^*\,|V^* = \pi\,|V^* : V^* \to U^*$ is a topological mapping.

Remark. In view of condition 2. Y inherits all the local properties of X. Consequently, although a coverin space of an arbitrary topological space may seem to be complicated, it will not be however very difficult to deal with some type of *regular* covering spaces over a complex *analytic* manifold X. X, just because of this inheritance (see theorem 2.5).

Definition 2.3. A corvering space is said to be regular [6] if there exists a continuation along any arc γ of X and from any point over the initial point of γ.

Definition 2.4. A cover transformation of a covering space. (Y, π) is a fiber preserving topological mapping of Y onto itself.

It is clear that the cover transformations form a group T.

As a consequence, a regular covering space of X can also be characterized by the property:

If σ_1, σ_2 are any two points on $\pi^{-1}(x)$ then there is an element $t \in$ T such that $t\,(\sigma_1) = \sigma_2$. [7].

Now, let F be the fundamental group of X. Let N^*, N be two normal subgroups of F defining the regular covering spaces Y^*, Y respectively. Then

Theorem 2.3. (Y^*,π^*) is a subcovering of (Y, π) if and only if N $\subset N^*$.

Proof. We follow the idea in [6]. Suppose that (Y^*, π^*) is a subcovering of (Y, π). To $\gamma \in N$ there corresponds a closed curve $\tilde{\gamma}$ on Y, say from $0 \in Y$, $0 \in \pi^{-1}$ (o), $o \in X$. There is a t which transforms $\tilde{\gamma}$ into a closed curve from $0^* \in Y^*$, $0^* \in \pi^{-1}$ (o), $0^* = t$ (0). Since by hypothesis π^{*-1} (x) $\subset \pi^{-1}$ (x) over any point x $\in$ X, this closed curve coincides with a closed curve $\tilde{\gamma}^*$ on Y^* from 0^* and whose projection by π^* is of course identical with γ. I. e., $\underset{\pi}{И} = \underset{\pi}{И}^*$ ot. Hence $\gamma \in N^*$ and so $N \subset N^*$.

Suppose now $N \subset N^*$. Let $\sigma \in Y, \sigma \in \pi^{-1}(x)$. Join 0 to σ by $\tilde{\gamma}$, determine the projection $\gamma = \pi(\tilde{\gamma}) \subset N$ and construct the continuation $\tilde{\gamma}^*$ on Y^* along $\gamma \subset N^*$ from $0^* \in \pi^{*-1}$ (o) $= \pi^{-1}(o)$. It follows that the terminal point σ^* of $\tilde{\gamma}^*$ is on π^{-1} (x) whose projection by $\pi^* = \pi | Y^*$ is x. Hence (Y^*, π^*) is a subcovering of (Y, π).

From there on we shall use the notation D for a normal subgroup of F such that F / D is commutative. By the Criterion every D contains the commutator subgroup [F, F] of F. *With respect to these D's we have*

Theorem 2.4. Let X be a Haudsdorff space with fundamental group $F \neq 1$. Then the regular covering space defined by [F, F] is maximal, i.e., is the strongest.

Proof. Theorem 2.3 shows that the ordering of regular covering spaces by inclusion is isomorphic with the ordering of the corresponding normal subgroups by inclusion. Moreover, each chain of regular covering spaces has an upper bound. By Zorn's lemma there is a maximal regular covering space of X that corresponds to [F.,F].

Furthermore, theorem 1.1 or the fundamental theorem [3] asserts that

Theorem 2.5. If X is a connected complex *analytic* manifold with fundamental group $F \neq 1$, then the maximal regular covering space of X defined by [F, F] is just the restricted analytic sheaf A, i.e., the sheaf of germs of the totality of holomorphic functions A (X) on X.

Proof. Since the detalis of the proof may be found in [3] we shall not repeat the proof here but only recall the basic idea. Let $\{U_\alpha, z_\alpha\}$, $\alpha \in I$, be an atlas of X. Then each point σ of the regular covering space defined by [F, F] inherits the local property of U_α, i.e., to σ is associated a corresponding local parameter z_α. If follows that each σ may be repre-

sented by a covergent power series $e = e_{x,\alpha}$ in z_α, say in U_α about $\pi(\sigma) = x \in U_\alpha$, and conversely. The representation $\varphi(\sigma) = e$ is ofcourse an isomorphism. For, it is one to one and on each fibre $\varphi(\sigma_1+\sigma_2) = e_1+e_2 = \varphi(\sigma_1) + \varphi(\sigma_2)$ The totality of these convergent power series defines the totality $A(X)$ of holomorphic functions on X which in turn defines the restricted analytic sheaf A [1]. Namely A is the disjoint union of the stalks (fibres) $\pi^{-1}(x)$ with $\pi^{-1}(x) \simeq F/(F,F)$ for each x.

In short, the regular covering space $\tilde{A}$ defined by $[F, F]$ and represented bv φ is homeomorphically isomorphic, hence identical, to A and is itself analytic complex manifold of dimension n with the projection map $\pi: \tilde{A} \to X$ holomorphic. In general F is not Abelian. Else A is the universal covering space of X.

Important Remark. The proof of the fundamental theorem is evident. Indeed, by hypothesis $\tilde{A}$ and A having the same fundamental group $[F,F]$ are isomorphically equivalent, i.e., there is a topological fibre (stalk) preserving mapping between them (see [1], p. 44, corollary). Actually, by construction, A is a complex analytic manifold, and the isomorphism between the fibre $\pi^{-1}(x)$ and the stalk $\pi_A^{-1}(x)$ issuint from an arbitrary point $x \in X$ makes the representation of $\sigma \in \pi^{-1}(x)$ by the corresponding convergent power series $e \in \pi_A^{-1}(x)$ evident.

Moreover, if $U = U(x) \subset U_\alpha$ is an open neighborhood, then the diagram defined by $e = \varphi(\sigma)$ and $e = \pi_A^{-1}o\pi(\sigma)$ is commutative and topological on U. It follows immediately that φ defines a one to one mapping between $\tilde{A}$ and A which in view of the diagram is easily seen to be an open mapping and therefore topological.

In fact, the corollary holds for the general case in which X is a connected locally arcwise connected Hausdorff space and the regular covering spaces have the same fundamental group. For, then the groups of covering transformations are isomorphic. In particular, the fibres issuing from each point are isomorphic. This yields a one to one correspondence between the points of the covering spaces. As before, this mapping is easily seen to be an open mapping. Hence it is topological. As a conclusion two regular covering spaces with the same fundamental group can always be identified. Otherwise stated,

Uniqueness Theorem. A regular covering space is uniquely determined up to an isomorphism.

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Proof. Let (Y,π), (Y',π') two regular covering spaces corresponding to a normal subgroup $N \subset F$. The one to one correspondence between the points $\bar{x} \in Y$, $\bar{x}' \in Y'$ follows directly from the deceomposition of F into the cosets $N\{a_1\}$ where a_i is a closed arc at x in x. Over ai lie the arcs joining x to $\sigma \in \pi^{-1}(x)$ and x to $\sigma' \in \pi'^{-1}(x)$ in Y, Y' respectively. Both arcs correspond to the element of the group F / N determined by a_1 [1]. F / N is isomorphic to the groups of cover transformations of Y, Y' respectively. As to the homeomorphism between Y,Y' it follows from the topological commutaıive diagrams.

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ÖZET

Bu makalede, tahditli demetler için bir kriter ile regular alt örtü uzaylar için bir gerek ve yeter şart verilmiştir.