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On Homology Covering Space and Sheaf Associated To The Homology Group

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On Homology Covering Space and Sheaf Associated To The Homology Group

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SUMMARY

In this paper we consider a connected and locally arcwise connected Hausdorff space X. If N is a normal subgroup of the fundamental group F of X such that F / N is Abelian then it is shown that the normal covering space determined by N is the sheaf of the additive groups isomorphic to F / N at each point $x \in X$ as x runs through X, and conversely. It turns out that if, in particular, X is an analytic manifold of dimension n, then the homology covering space $\tilde{A}$ determined by the homology group F / [F,F] is itself an analytic complex manifold of dimension n with the projection map $\pi : \tilde{A} \to X$ holomorphic. It follows at once that $\tilde{A}$ is the sheaf A of germs of the totality of holomorphic functions A (X) on X.

1. NORMAL COVERING SPACES

In this work the base space X will be connected and localy arcwwise connected Hausdorf space.

It is known that if a topological space is connected and locally arcwise connected then it is also arcwwise connected.

Definition 1.1. Let Y be a topollogical space. $\pi: Y \to X$ is a covering space of X if every $x \in X$ has an open neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open sets s_i in Y, each of which is mapped homeomorphically onto U by π. Such U are said to be evenly covered, and the s_i are called sheets over U. [1].

As immediate consequences:

1. The Fibre π^{-1} (x) over any point $x \in X$ is *discrete.*

2. π is local homeomorphic.

3. π is continuous.

π is called a *projection*. The covering space is usually denoted by (Y, π) or simply by Y if no confusion arizes.

Definition 1.2. A covering space is said to be *regular* [2] if there exists a continuation along any arc γ of X and from any point over the initial point of γ.

Definition. 1.3. *A cover transformation* of a covering space (Y,π) is a fibre preserving topological mapping of Y onto itself.

It is clear that the cover transformations from a *group* T.

As a consequence, a regular covering space of X can also be characterized by the property:

If σ_1, σ_1 are any two points on π^{-1} (x) then there is an element $t \in T$ such that $t(\sigma_1) = \sigma_2$.

Let $F = \pi_1 (X, o)$, $o \in X$, be the fundamental group of X and $N \subset F$ a normal subgroup. Then to N corresponds a regular covering space of X which we denote again by Y. We have for a regular covering space Y [3].

Theorem 1.1. The group T of cover transformations of Y is, isomorphic with the quotient group F / N.

Proof. Let $N \{a_{11}\}$, $N \{a_{12}\}$,... be the cosets of F / N. Here a_{1i} is a closed arc in X at o, $\{a_{1i}\}$ is the class of all closed arcs at o homotopic to a_{1i}, with $\{ a_{1i}\} = \{1\}$. We have the decomposition.

$$F = N \{a_{11}\} + N \{a_{12}\} + \dots$$

and the cosets are in one to one correspondence with the points $\tilde{O}_1$, $\tilde{O}_2$, ... lying in Y over o such that to each point A_1 there corresponds a a covering transformation t_{1i} transforming $\tilde{O}_1$ to $\tilde{O}_i$. Thus there is a one correspondence between the cosets $N\{a_{1i}\}$ and the covering transformations t_{1i}, i.e.,

$$N\{a_{1i}\} \leftrightarrow t_{1i}.$$

We maintain that the correspondence is a an isomorphism between the quotient group F / N and the group T of covering transformations.

Namely, to the product N $\{a_{1i}\}$. N $\{a_{1j}\}$ = N $\{a_{1k}\}$ there corresponds the product $t_{1i}\, t_{1j} = t_{1k}$ of the corresponding covering transformations. Denote by $\tilde{a}_{1i}$ the arc lying in Y over a_{1i} and joining A_1 to A_i. Consider the product of the two cosets N$\{a_{1i}\}$. N $\{a_{1j}\}$ = N $\{a_{1i}\}\{a_{1j}\}$ = N $\{a_{1,k}\}$. One can reach the point $\tilde{O}_k$ lying over o first going from $\tilde{O}_1$ to $\tilde{O}_i$ along an arc $\tilde{a}_{1i}$ and then arrive at $\tilde{O}_k$ from $\tilde{O}_i$ going along the arc $\tilde{a}_{ik}$ lying above a_{1j} Since $\tilde{a}_{ik}$ and $\tilde{a}_{1j}$ lie above the same ground closed arc a_{1j}, it follows that $\tilde{a}_{1k}$ is the image of $\tilde{a}_{1j}$ under the covering transformation t_{1i}. Thus t_{i1} transforms the point $\tilde{O}_j$ to $\tilde{O}_k$ and therefore the covering transformation D_{1i} D_{1j} transforms the point $\tilde{O}_1$ to $\tilde{O}_k$. Hence $D_{1i}\ D_{1j} = D_{1k}$.

From here on we shall assume that F / N is Abelian. There is the following important fact not mentioned in the literature [3]. The points $\tilde{O}_i$ on the fibre π^{-1} (o) form an addivite group. Define the arc $\tilde{a}_{1i}$ by the continuous mapping f_i (θ) which maps the unit interval I = [0,1] into Y. We define $f_1 + f_2 : I \to fY_2$ by setting

$$(f_1 + f_2)\ (\theta) = f_1\ (\theta) + f_2\ (\theta).$$

for every θ of I. One can easily verify that under this operation the f_i's and so the $\tilde{a}_{1i}$'s form a gropup. In particular the $\tilde{O}_1$'s which correspond to the point $1 \in I$ form an Abelian group. It as again easy to verify that the correspondence

$$\tilde{O}_i \leftrightarrow t_{1i}$$

is an isomorphism and therefore π^{-1} (o) is isomorphic to the quotient group F / N. Since o is arbitrary it follows that for each $X \varepsilon x$, π^{-1} (x) is isomorphic to F / N. Hence we may state.

Theorem 1.2. If N is a normal subgroup such that F / N is Abelian then the fibres are Abelian groups all isomorphic to the quotient group F / N.

Definition 1.4. Denote π^{-1} (x) by Y_x, called a stalk, and note that $Y_x \subset Y$, $\bigvee_{x \varepsilon X} Y_x = Y$. Then together with definition 1.1, the triple (π, Y,X) is also called a sheaf of the Abelian groups Y_x over X.

As a consequence, $(\sigma_1, \sigma_2) \to \sigma_1 + \sigma_2$ defined on the set $Y \oplus Y$ of pairs (σ_1, σ_2) such that σ_1, σ_2 belong to the same stalk, is a continouos mapping of the subset $Y \oplus Y$ of Y x Y into Y. [4]

For simplicity, We again call Y the sheaf (π, Y, X).

Definition 1.5. Let U $\subset$ X be open. A section of Y over U is a continuous map s of U into Y into Y that π os is the identity mapping.

A section over X is defined in a similar way [4].

π : S (U)$\rightarrow$ U is topological and s $=$ (π / s (U))$^{-1}$. [5].

Denote by Γ (U,Y) the additive group of all section of Y over U. Again, if N is a normal subgroup such that F / N is Abelian, then

Theorem 1.3. Γ (U,Y) $\cong$ F / N.

In particular, Γ (X, Y) $\cong$ F / N.

Definition 1.6. A regular covering space wich corresponds to a normal subgroup N is called a normal covering space. [2].

Definition 1.7. The normal covering space determined by the commutator subgroup [F, F] is called the homology covering space.

The quotient group F / [F,F] is called the homology group of X. It may be defined at each point xεX since the fundemental group π_1, (X, x) at each point x is isomorphic to F.

The homology group is of course Abelian. Since it is defined at each point xεX, the homology covering space may also be called the sheaf of homology groups at X. It is the disjoint union of homology groups at x.

For normal covering spaces (Y, π) it is shown that (Y*, π*) is a subcovering of (Y,π) if and only if N $\subset$N*. And that F / N is Abelian if and only if N $\supset$ [F, F]. [6].

It then follows by Zorn's lemma that the homology covering space is maximal.

Let us now consider the particular case in which X is an analytic complex maniold of dimension n. Let [F, F] be the commutator subgroup of F. It is the smallest normal subgroup such that F / [F, F)] is commutative. Then the holmology covering space $\tilde{A}$ of X is itself an analytic complex manifold of dimension n, and the projection mapmapping π : A $\rightarrow$ X is of course holomorphic. We may state [6] [7], Fundamental Theorem. Let X be a connected complex analytic manifold of dimension n with fundamental gruop $F \neq 1$. Then [F,F] determines completely the vector space A (X) of holomorphic functions on X and thereby the sheaf A of holomorphic functions A(X) on X. [8].

2. *Sheaf of Homology groups at x.* We shall now conversely construch directly the sheaf of holology groups at x, over X as follows and then show that it is the homology covering space of X with fundamental group [F, F]. The problem is similar to the one treated in the paper [8].

Let [F, F] be the commutator subgroup of F. The quotient group F / [F, F] is Abelian. At each point $x \varepsilon X$ we can form the quotient group $Y_x = (F / [F, F])_x$ which is isomorphic to F / [F,F] formed at an arbitrary point o of X. Indeed, X being arcwise connected π_1 (X, x) $\cong \pi_1$ (X, o) and so $Y_0 \cong Y_x$ at each $x \varepsilon X$. Each element σ of Y_x is a residu class, that is a coset of $(F / [F,F])_x$.

The disjoint union

$$Y = \bigvee_{x \in X} Y_x$$

is a set over X with a natural projection

$$\pi : Y \to X$$

which sends all the elements of Y_x to x.

If $\sigma_0 \varepsilon Y$ is an element, then σ_0 belongs to some Y_{x_0} and maps σ_0 onto X_0.

We introduce on Y a natural topology as follows. By hypothesis there exists an arcwise connected open neighborhoodd $U = U(X_0) \subset X$ such that to each $x \varepsilon U$ there corresponds a Y_x obtained by connecting x to x_0 by an arc lying in U. This induces a mapping.

$$s : U \to Y$$

defined by $s(x) = \sigma \varepsilon Y_x \subset Y$ such that $\pi \circ s = 1_u$ and $s(x_0) = \sigma_0 \in s(U) \subset Y$.

All such sets s(U) form a system of neighborhoods of σ_0 which induces a topology in Y.

Hence Y is a sheaf over X called the sheaf of the homology groups at x as x runs through X.

In this topology s is continuous. For, let $\sigma \varepsilon Y$. By construction of Y there exists $x \varepsilon X$ such that $\sigma \varepsilon Y_x$. If s (U) is any neighborhood of σ, then there exists V (x) $\subset$ U such that s(V) $\subset$ s (U). Similarly π is continuous in this topology. s is called a section over U, and the totality of sections over U is denoted by Γ (U, Y) which is of course an Abelian group isomorphic to F / [F, F). Moreover, π: s(U) $\rightarrow$ U is topological. [5].

Consequently, every $x \varepsilon X$ has an open neighborhood U such that π^{-1} (U) is a disjoint union of open sets s_i in Y, each of which is mapped homeomorphically onto U by π. Hence π: Y $\rightarrow$ X is at the same time a covering space of X.

One can prove as in [8] that Y is a regular covering space of X.

Our next step is to show that Y is a normal covering space determined by [F, F]. More presicely, we shall show that [F, F] is the fundamental group of Y.

Theorem 2.1. [F, F] is the fundamental group of Y.

Proof. It is similar to the proof of theorem 1.1. The cosets of F– [F, F] are of the form.

$$[F,F]\ \{a_{11})\},\ \{F,F]\ \{a_{12}\},\ldots$$

where $\{a_{1i}\}$ is the class of all closed arcs, say at o, homotopic to a_{1i}. In particular, $\{a_{11}\} = \{1\}$. We have as before the decomposition

$$F = [F,F]\ \{a_{11}\} + [F,F]\ \{a_{12}\} + \ldots$$

and the cosets are in one to one correspondence with the points $\tilde{O}_1$, $\tilde{O}_2$,... lying in Y over o. We recall that o ε X is arbitrary but fixed. In other words they form the stalk $Y_0 \subset Y$.

Let a_{1i} be the closed arc in X and $\tilde{a}_{1i}$ the arc lying over a_{1i} with initial point $\tilde{O}_1$ over o. The arc $\tilde{a}_{1i}$ joins $\tilde{O}_1$ to $\tilde{O}_i$, while o_i determines the unique section s_i (X) through $\tilde{O}_i$, and $\{a_{1i}\}[F,F]\{a_{1i}\}^{-1} = [F,F]$ is lifted into s_i (X) if the chosen initial point in Y is $\tilde{O}_i$. This induces a cover transformation of Y into itself taking $\tilde{O}_1$ into $\tilde{O}_1$, and the zero section s_1 (X) into s_i (X). This proves that the fundamental group of Y is [F,F].

3. *Čech Cohomology*. This section is refered to [9] with the obvious corresponding modifications.

In particular, if H^o (X,Y) is the 0–th cohomology group of X with values in Y, then.

$$H^o (X,Y) \cong \Gamma (X,Y).$$

Since the covering transformations group T of the homology covering space Y of X is isomorphic to the homology group F / [F,F] which in turn is isomorpic to Γ (X,Y) it follows that.

Theorem 3.1. The homology froup F / [F,F] of X is isomorphic to the 0-th cohomology. group H^0 (X,Y) of X with values in Y. Namely,

$$F / [F,F] \cong H^o (X,Y).$$

In particular, if X is an analytic complex manifold then the homology covering pase of X is just the sheaf A of germs of the totality A (X) of holomorphic funditons on X. The group of sections in A over X, Γ (X, A) is isomorphic to yhe group A (X) and Theorem 3 / 1 reads in this case [8].

Theorem 3.2. If X is an analytic comple manifold then the homology group F / [F,F] is on the one hand isomorphic to the 0-th cohomology group H^0 (X,A) of X with values A and the other hand to the additive group A(X) of holomorphic funcions on X. Namely,

$$A(X) \cong \Gamma (X,A) \cong F / [F F] \cong H^o (X,Y) \cong T.$$

Since the set of all conjugate classes [F, F] $\{a_{1i}\}$ forms also a group isomorphic to the quotient group F/[F, F], [10] we have at any base point x, the decomposition

$$F = [F, F] + [F, F] \{a_{12}\} + \dots .$$

It follows that T is now isomorphic to the group of biholomorphic self maps of X.

Finally we remark that in the decomposition of F into the cosets [F, F] $\{a_{1i}\}$ each homotopy class $\{a_{1i}\}$ is a representatif, and therefors the set of representatives $\{1\}$, $\{a_{12}\}$, ... forms an Abelian group isomorphic to F/[F, F] called the Abelianized fundamental group. For, the one to one correspondence

$$\{a_{1i}\} \rightarrow [F, F] \{a_{1i}\}$$

implies

$$\{a_{1i}\} \{a_{1j}\} \rightarrow [F, F] \{a_{1i}\} \{a_{1j}\} = [F, F] \{a_{1i}\}.[F,F] \{a_{1j}\}.$$

The above Abelian group will be called the first Homology group.

We have seen that this correspondence generates the homology covering space with fundamental group [F, F] and thereby gives a procedure of Abelianizing F by just forming the quotient group F/[F, F] of the two fundamental groups.

REFERENCES

[1]- **M.J. Greenberg,** Lectures On Algebraic Topology, W.A. Benjamin İnc. 1967.

[2]- **L.V. Ahlofors and L. Sorio,** Riemann Surfaces, Princeton University Press, Princeton, New Jersey, 1960.

[3]- **Sifert and Threlfall,** A Textbook of Topology, translated by M.A. Goldman, Acemeic Press, 1980

[4]- **L. Bers,** Introduction to Several Complex Variables, Courant Institute of Mathematical Sciences, New York University, 1964.

[5]- **H.Grauert K. Fritzsche,** Several Complex Variables, Springer- Verlag, 1976.

[6]- **C. Uluçay,** A Criterion For Restricted Ideal Sheaves Comm. Fac. Sc. Ankara Univ. 33 (1984)15–19.

[7]- ———, Homology Group and Generalized Riemann-Roch Theorem Comm. Fac. Sci. Univ. Ankara. 32 (1982), 49–53

[8]- ———, On the Homology Group of The Complex Analytic Manifolds, Com. Fac. Sc. Ankara Un. Vol. 30, pp. 37-44, 1972.

[9]- ———, The Cousin and Poincare Problems, Com. Fac. Sc. Ankara Un. Vol. 30, 9pp. 27–36, 1982.

[10]- ———, Algebraic Structure of The Restricted Analytic Sheaves and Restricted Ideal Shaves, Jour. Fac. Sc. Karadeniz Un. Pp. 10-14, 1983, Trabzon.

ÖZET

Bu makalede, F/N aditif olmak üzere, gösteriliyor ki N F normal altgruplarına tekabül eden X Hausdorf uzayının normal örtü uzayları X üzerinde demettirler. Özellikle Homoloji Örtü uzayları üzerinde duruluyor.