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by

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Some Theorems On The Sheaf Of The Fundamental Groups.

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SUMMARY

Let X be a locally arcwise connected topological space. In paper [1], we constructed "the sheaf of the fundamental groups" on X and gave some characterizations. In paper [2], we first gave some characterizations which are the converses of the characterizations in paper [1] and obtained some results related with the group of sections. In this paper, we first construct the sheaf $H^* = H_1 \oplus H_2$ by defining "the Generalized Whitney Sum" and show that the sheaf H^* is isomorphic to the sheaf $H_1 x H_2$ which is direct product of the sheaves H_1 and H_2 on X_1 and X_2, respectively. Finally, we prove that, if (X_1, H_1) and (X_2, H_2) are any two pairs, then H^* is isomorphic to the pairs $(H, X_1 x X_2)$.

1. INTRODUCTION.

Let X be a locally arcwise connected topological space and $\pi_1(X,x)$ be the fundamental group at x for any point $x \in X$, then the disjoint union $H = \underset{x \in X}{V} \pi_1(X,x)$ is a set over X with natural projection $\varphi: H \rightarrow X$ mapping each $\sigma_x = [\alpha]_x$ onto the base point.

We introduced on H a natural topology as follows [1]:

Let $x \in X$ be an arbitrary fixed point. Then there exists an arcwise connected open neighborhood $U = U(x)$. If $[\alpha]_x \in \pi_1(X,x)$ is an arbitrary fixed element and $y \in U$ is any point, then there exists an element $[\beta]_y \in \pi_1(X, y)$ which uniquely corresponds to $[\alpha]_x$, since $\pi_1(X,x) \cong \pi_1(X, y)$. Therefore we can define a mapping $s: U \rightarrow H$ with $s(y) = [\beta]_y$ for any $y \in U$ such that $\varphi \circ s = 1_U$ and $s(x) = [\alpha]_x \in s(U) \subset H$. It should be noticed that s is related with the homotopy class $[\alpha]_x$ but does not related with the homotopy class of δ which is an arc with initial point x and terminal point y, since the homotopy class $[\delta]$ is same fixed for every s [6,7].

For each $x \in X$ all such sets $s(U)$ form a system of neighborhood of $[\alpha]_x \in H$ which induces a topology in H. In this topology s is continuous and φ is a locally topological maping. s is called a section over U and the totality of sections over U is denoted by $\Gamma(U,H)$. For the definition of a section over any open set $W \subset X$ see [1]. In paper [1] we proved that, if $W \subset X$ is any open set, then $\Gamma(W, H)$ is a group. Therefore the operation of multiplication on each stalk $\odot : H \oplus H \to H$ is continuous and so H is a sheaf with an algebraic structure. We call this sheaf as "The Sheaf of the Fundamental Groups" [3,4].

2. SOME THEOREMS ON THE SHEAF OF THE FUNDAMENTAL GROUPS.

Let the pairs (X_1, H_1), (X_2,H_2) be given. Consider the sets of the sections $\Gamma_1(W_1,H_1)$ and $\Gamma_2(W_2,H_2)$ being $W_1 \subset X_1$ and $W_2 \subset X_2$ are open sets. Let $M_W = \Gamma_1(W_1,H_1) \times \Gamma_2(W_2,H_2)$ such that $W = W_1 \times W_2 \subset X_1 x X_2$. For an element $s = (s_1,s_2) \in M_W$ and an open set $V \subset W$ ($V = V_1 x V_2$; $V_1 \subset W_1$, $V_2 \subset W_2$ are open sets) let $\gamma_{W,V}(s) = \gamma_{W,V}((s_1.s_2)) = (\gamma_{W_1,V_1}(s_1), \gamma_{W_2,V_2}(s_2)) = (s_1|V_1, s_2|V_2)$. Then, the system $\{X_1xX_2, M_W, \gamma_{W,V}\}$ is a presheaf. Thus, forming inductive limit, a sheaf is obtained from the pre-sheaf $\{X_1 xX_2, M_W, \gamma_{W,V}\}$ [3]. Now, we can give the following definition.

Definition 2.1. The sheaf which is obtained from the pre-sheaf $\{X_1xX_2, M_W, \gamma_{W,V}\}$ by forming inductive limit is called "the Generalized Whitney Sum" of the sheaves H_1 and H_2, and denoted by $H^* = H_1 \oplus H_2$.

Let us now show that, the stalk $H^*_{(x_1,x_2)}$ has a group structure for each $(x_1,x_2) \in X_1xX_2$. In fact, if $H^*_{(x_1,x_2)} = \{(W, (s_1, s_2))_{(x_1,x_2)} : W = W((x_1,x_2)) \subset X_1xX_2$ is an open$\}$ and $(W, (s_1, s_2))_{(x_1,x_2)}, (W', (s'_1, s'_2))_{(x_1,x_2)} \in H^*_{(x_1,x_2)}$ are any two elements, then let $(W,(s_1,s_2))_{(x_1,x_2)} . (W', (s'_1,s'_2))_{(x_1,x_2)} = (W'', (s_1 . s'_1, s_2 . s'_2))_{(x_1,x_2)}$, where $W'' = W''_1 \times W''_2$ and $W''_1 = W_1 \cap W'_1$, $W''_2 = W_2 \cap W'_2$. That is, $(W, (s_1,s_2))_{(x_1,x_2)} . (W', (s'_1,s'_2))_{(x_1,x_2)} = (W'', (\gamma_{W_1,W_1''}(s_1) . \gamma_{W_2,W_2''}(s'_1), \gamma_{W_2,W_2''}(s_2) . \gamma_{W_2',W_2''}(s_2))$. Thus, $s_1.s_1' \in \Gamma_1(W''_1, H_1)$, $s_2.s_2' \in \Gamma_2(W''_2,H_2)$. Therefore, the operation of multiplication defined above is well-defined and closed in $H^*_{(x_1,x_2)}$. It is esialy established that $H^*_{(x_1,x_2)}$ is a group with respect to this operation of multiplication.

On the other hand, if we define an other operation of multiplication on $(H_1)_{x_1}$ x $(H_2)_{x_2}$ as $(\sigma_1, \sigma_2).(\sigma'_1, \sigma'_2) = (\sigma_1.\sigma'_2, \sigma_2.'_2)$ for any two elements $(\sigma_1, \sigma_2), (\sigma'_1, \sigma'_2) \in (H_1)_{x_1}$ x $(H_2)_{x_2}$, then the defined operation of multiplication is well-defined and closed in $(H_1)_{x_1}$ x $(H_2)_{x_2}$, since $\sigma_1.\sigma'_1 \in (H_1)_{x_1}$, $\sigma_2.\sigma'_2 \in (H_2)_{x_2}$. It is esialy shown that $(H_1)_{x_1}$ x $(H_2)_{x_2}$ is a group with this operation of multiplication [4].

We can now give the following theorem.

Theorem 2.1. Let the pairs (X_1, H_1) and (X_2, H_2) be given such that $H^* = H_1 \oplus H_2$. Then, for each $(x_1, x_2) \in X_1 x X_2$ the mapping $k:H^*_{(x_1,x_2)} \to (H_1)_{x_1}$ x $(H_2)_{x_2}$ defined by $(W,(s_1,s_2))_{(x_1,x_2)} \to (s_1(x_1), s_2(x_2))$ is an isomorphism.

Proof 1. Let $(x_1, x_2) \in X_1 x X_2$ any point and W, W' be any two open neighborhoods of (x_1, x_2). It is known that; $(W, s = (s_1,s_2)) \overset{(x_1,x_2)}{\sim} (W', s' = (s'_1, s'_2))$ if and only if there exists a neighborhood $V((x_1,x_2)) \subset W \cap W'$ such that $s|V = s'|V$. This is also equivalent to the statement $s(x) = s'(x)$. Therefore k is well-defined and injective.

2. Let $\sigma = (\sigma_1, \sigma_2) \in (H_1)_{x_1}$ x $(H_2)_{x_2}$. Then, there exists the open neighborbhoods $W_1 \subset X_1$, $W_2 \subset X_2$ such that $s_1(x_1) = \sigma_1$, $s_2(x_2) = \sigma_2$ for the sections $s_1 \in \Gamma_1(W_1, H_1)$, $s_2 \in \Gamma_2(W_2, H_2)$. Thus, $W = W_1 x W_2$ is open neighborhood of (x_1, x_2) and $s(x_1, x_2) = (s_1(x_1), s_2(x_2))$ for $s = (s_1, s_2)$, i.e., $s \in M_W$. Hence γs is a section over W such that $\gamma s(x) = \gamma s((x_1, x_2)) = (W, s = (s_1, s_2))_{(x_1,x_2)}$ and $k((W, s = (s_1,s_2))_{(x_1,x_2)}) = (s_1(x_1), s_2(x_2)) = \sigma$. Therefore k is a surjective mapping.

3. For any elements $(W, (s_1,s_2)), (W', (s'_1,s'_2)) \in H^*_{(x_1,x_2)}$, $k((W,(s_1, s_2)) . (W', (s'_1, s'_2)) = k(W'', (s_1.s'_1, s_2.s'_2)) = (s_1(x_1) . s'(x_1), s_2(x_2).s'_2(x_2)) = (s_1(x_1), s_2(x_2)).(s'_1(x_1), s'_2(x_2)) = k((W, (s_1, s_2)))$. $k((W', (s'_1, s'_2))$. Thus, k is a homomorphism.

Therefore k is an isomorphism.

From now on, we identify $H^*_{(x_1,x_2)}$ with $(H_1)_{x_1}$ x $(H_2)_{x_2}$.

Now, let the pairs (X_1,H_1), (X_2, H_2) be given. Then, $H_1 = \bigvee_{x_1 \in X_1} (H_1)_{x_1}$, $H_2 = \bigvee_{x_2 \in X_2} (H_2)_{x_2}$. Hence, $H_1 x H_2 = \bigvee_{(x_1,x_2) \in X_1 x X_2} (H_1)_{x_1} x (H_2)_{x_2}$.

Thus, H_1xH_2 is also a set over the topological space X_1xX_2. Moreover, H_1xH_2 is also a topological space, since H_1,H_2 are topological spaces. Let us now define a mapping $\Phi:H_1xH_2 \to X_1xX_2$ as follows:

If $(\sigma_1, \sigma_2) \in H_1xH_2$, then let $\Phi((\sigma_1, \sigma_2)) = (\varphi_1(\sigma_1), \varphi_2(\sigma_2)) = (x_1,x_2) \in X_1 x X_2$.

We assert that $\Phi = (\varphi_1, \varphi_2)$ is a locally topological mapping. In fact, if $(\sigma_1, \sigma_2) \in H_1xH_2$, then $\Phi((\sigma_1, \sigma_2)) = (\varphi_1 (\sigma_1), \varphi_2(\sigma_2)) = (x_1, x_2)$. Since the mappings $\varphi_1 : H_1 \to X_1$, $\varphi_2 : H_2 \to X_2$ are locally topological, there exists open neighborhoods $U_1(\sigma_1) \subset H_1$, $W_1(x_1) \subset X_1$, $U_2(\sigma_2) \subset H_2$, $W_2(x_2) \subset X_2$ such that $\varphi_1|U_1:U_1 \to W_1$, $\varphi_2|U_2:U_2 \to W_2$ are topological. Therefore $U(\sigma_1,\sigma_2) = U_1(\sigma_1) xU_2(\sigma_2)$ and $W = W_1(x_1)x W_2(x_2)$ are open neighborhoods of the points (σ_1, σ_2) and (x_1, x_2), respectively. Finally, it is clearly seen that $\Phi|U:U \to W$ is a topological mapping.

Thus, (H_1xH_2, Φ) is a sheaf over X_1xX_2. Moreover, $\Gamma(W,H_1 x H_2)$ is a group, for any open $W \subset X_1xX_2$. Hence, the operation of multiplication is continouous on each stalk with respect to the topology of $H_1x H_2$. Therefore, H_1xH_2 is a sheaf with algebraic structure [4].

Definition 2.2. Let the pairs (X_1, H_1) and (X_2, H_2) be given. Then the sheaf H_1xH_2 is called the Direct Sum of the sheaves H_1 and H_2.

We can now give the following theorem.

Theorem 2.2. Let the pairs (X_1, H_1), (X_2, H_2) be given. Then the sheaves $H^* = H_1 \oplus H_2$ and H_1xH_2 are isomorphic.

Proof. Let us assume that, $H^* = (H^*, \varphi^*)$ and $H_1xH_2 = (H_1xH_2, \Phi = (\varphi_1, \varphi_2))$.

First show that the mapping $K:H^* \to H_1xH_2$ defined by $(W, (s_1, s_2))_{(x_1, x_2)} \to (s_1 (x_1), s_2(x_2))$ is continuous. Now, let $U \subset K(H^*)$ is an open, i.e., $U = U_1xU_2$, $U_1 = \bigcup_{i \in I} s_i^1(W_i)$, $U_2 = \bigcup_{j \in J} s_j^2(V_j)$. Hence $U_1 = s_1 (W)$, $U_2 = s_2 (V)$ and $U = s_1 (W) x s_2 (V)$, where $s_1 \in \Gamma(W,H_1)$, $s_2 \in \Gamma(V,H_2)$ and $W = \bigcup_{i \in I} W_i$, $V = \bigcup_{j \in J} V_j$ [1]. So $\Phi(U) = WxV$.

Show that, $K^{-1}(U) \subset H^*$ is an open set. In fact, if $\sigma^* \in K^{-1}(U)$, then there exists an element $\sigma \in U$ such that $K(\sigma^*) = \sigma$. Therefore, if $\sigma^* = (\Phi (U), (s_1, s_2))_{(x_1,x_2)}$, then $K(\sigma^*) = (s_1 (x_1), s_2(x_2))_{(x_1,x_2)} \in U$ and $\Phi(K(\sigma^*)) = (x_1, x_2) \in \Phi(U)$. Thus, there exists a section $\gamma\ s =$

$\gamma(s_1, s_2)$: Φ (U) $\to$ H^* such that γs $((x_1, x_2))$ = $(\Phi(U), (s_1, s_2))_{(x_1,x_2)}$= $\sigma^* \in \gamma$ s $(\Phi(U))$. Therefore, K^{-1} (U) $\subset$ γ s $(\Phi(U))$, since σ^* is an arbitrary element. On the other hand, if $\sigma^{*\prime} \in \gamma$ s (Φ (U)), then $\sigma^{*\prime}$ = (Φ (U), $(s_1,s_2))_{(x_1',x_2')}$ and φ^* $(\sigma^{*\prime})$ = $(x'_1,x'_2) \in \Phi$ (U). So $(s_1 (x'_1)$, $s_2(x'_2)) \in$ U. However, K $(\sigma^{*\prime})$ = $(s_1 (x'_1), s_2 (x'_2))$. Therefore $\sigma^{*\prime} \in$ $K^{-1}(U)$ and γ s (Φ (U)) $\subset$ K^{-1} (U) $\subset$ H^* is an open set.

By Theorem 2.1., the mapping K is a bijection since K $|H^*_{(x1,x2)}$ = K, for each stalk $H^*_{(x_1,x_2)} \subset H^*$. On the other hand, K is a stalk preserving mapping, since (Φ oK) (σ^*) = φ^* (σ^*), for every $\sigma^* \in H^*$. Therefore K is a sheaf morphism. Moreover K^{-1} is continuous, since K is an open mapping.

Thus, the mapping K is a sheaf isomorphism. From now on, we identify H^* with H_1xH_2.

We can now state the following theorem.

Theorem 2.3. $\Gamma(W, H^*)$ is isomorphic to $\Gamma(W, H_1xH_2)$, for each open set $W \subset X_1xX_2$.

We can now give the following theorem.

Theorem 2.4. Let the pairs (X_1, H_1) and (X_2, H_2) be given. Then the mapping $P=(p^i, p_i)$ $(H_1xH_2, X_1xX_2) \Rightarrow (H_i, X_i)$ is a homomorphism, i=1, 2.

Proof. Let us first show that p^i is a stalk preserving mapping with respect to p_i. In fact, $(p_i \text{ o } \Phi)$ (σ_1, σ_2) = p_i $(\Phi(\sigma_1, \sigma_2))$ = p_i (x_1,x_2) = x_i and $(\varphi_i o p^i)$ (σ_1, σ_2) = φ_i $(p^i(\sigma_1, \sigma_2))$ = φ_i (σ_i) = x_i and so $p_i o \Phi$= $\varphi_i o p^i$, for each element $(\sigma_1, \sigma_2) \in H_1xH_2$.

p^i is a homomorphism on each stalk. Indeed, $p^i((\sigma_1, \sigma_2).(\sigma'_1, \sigma'_2))$ = p^i $((\sigma_1.\sigma'_1, \sigma_2.\sigma'_2))$ = $\sigma_i.\sigma'_i$, i=1, 2. However, $\sigma_i.\sigma_i'$ = p^i (σ_1, σ_2). p^i (σ'_1, σ'_2), i=1,2. Therefore, p^i is a homomorphism on each stalk.

The mappings p^i: $H_1xH_2 \to H_i$, p_i: $X_1xX_2 \to X_i$ are continuous, since they are projections.

Thus P is a homomorphism between the pairs (H_1xH_2,X_1xX_2) and (H_i, X_i), i=1,2.

We can state the following theorem.

Theorem 2.5. Let the pairs (X_1, H_1) and (X_2, H_2) be given. Then the mapping P^*: $(H^*, X_1xX_2) \Rightarrow (H_i,X_i)$ is a homomorphism.

We can now give the following theorem.

Theorem 2.6. Let the pairs (X_1, H_1) and (X_2, H_2) be given. Then the sheaf H constructed over X_1xX_2 and the sheaf H_1xH_2 are isomorphic.

Proof. It is known that the projections $p_1:X_1xX_2 \to X_1$ and $p_2: X_1xX_2 \to X_2$ are continuous. Let $\alpha:I \to X_1xX_2$ be a closed arc at (x_1, x_2) for any point $(x_1, x_2) \in X_1xX_2$. Then, the mappings $p_1 \circ \alpha:I \to X_1$, $p_2 \circ \alpha:I \to X_2$ are also continuous and closed arcs at $x_1 \in X_1$, $x_2 \in X_2$, respectively, since $p_1(\alpha(0)) = p_1(\alpha(1)) = x_1$, $p_2(\alpha(0)) = p_2(\alpha(1)) = x_2$.On the other hand, if α_1,α_2: $I \to X_1xX_2$ are closed arcs at (x_1, x_2) such that $\alpha_1 \sim \alpha_2$, then $p_1 \circ \alpha_1 \sim p_1 \circ \alpha_2$, and $p_2 \circ \alpha_1 \sim p_2 \circ \alpha_2$. Therefore, the correspondence $[\alpha]_{(x_1,x_2)} \to ([p_1 \circ \alpha]_{x_1}, [p_2 \circ \alpha]_{x_2})$ is well-defined, for an arbitrary fixed point $(x_1, x_2) \in X_1xX_2$. That is, to the element $[\alpha]$ there uniquely corresponds an element $([p_1 \circ \alpha]_{x_1}, [p_2 \circ \alpha]_{x_2})$. Since the point $(x_1,x_2) \in X_1xX_2$ is an arbitrary point we obtain a mapping $\Psi : H \to H_1xH_2$ such that $\Psi(\sigma) = \Psi([\alpha]) = ([p_1 \circ \alpha], [p_2 \circ \alpha])$, for any $\sigma \in H$.

Let us now show that Ψ is a sheaf isomorphism.

1. Ψ is a sheaf morphism. In fact, Ψ is a stalk preserving mapping, since $\Phi\ (=(\varphi_1, \varphi_2)) \circ \Psi = \varphi$. On the other hand, if $U \subset \Psi(H)$ is an open, then $\Psi^{-1}(U) \subset H$ is an open. Because if, $U \subset \Psi(H)$ is an open, then there exists the open sets U_1 and U_2 in H_1 and H_2 respectively, such that $U = U_1xU_2$. Therefore, $U_1 = s_1(W_1)$, $U_2 = s_2(W_2)$ for the sections $s_1 \in \Gamma(W_1,H_1)$ and $s_2 \in \Gamma(W_2, H_2)$. Thus, $U = s_1(W_1)\ xs_2(W_2)$ and $\Phi(U) = W_1xW_2$. However, $s(W_1xW_2) \subset H$ is an open for any section $s \in \Gamma(W_1xW_2, H)$. Let us now show that $\Psi^{-1}(U) = s(W_1xW_2)$ for a section $s(W_1xW_2)$.

(i) If $\sigma = [\alpha]_{(x_1,x_2)} \in \Psi^{-1}(U)$, then there is at least one element $\sigma^* \in U \in \Psi(\sigma) = \sigma^* = ([p_1 \circ \alpha]_{x_1}, [p_2 \circ \alpha]_{x_2})$. So, $\Phi(\sigma^*) = (x_1, x_2) \in W_1x W_2$ and there exists a section $s \in \Gamma(W_1xW_2,H) \in s((x_1,x_2)) = \sigma$. Therefore, $\sigma \in s(W_1xW_2)$ and $\Psi^{-1}(U) \subset s(W_1xW_2)$.

(ii) If $\sigma' \in s(W_1xW_2)$, then $\sigma' = [\alpha']_{(x_1',x_2')}$ and $\varphi((\sigma') = (x'_1,x'_2) \in W_1xW_2$. However, $\Psi(\sigma') = \Psi([\alpha']_{(x_1',x_2')}) = ([p_1 \circ \alpha']_{x_1}, [p_2 \circ \alpha']_{x_2})$ and $\Psi(\sigma') \in U$. Therefore, $\sigma' \in \Psi^{-1}(U)$ and $s(W_1xW_2) \subset \Psi^{-1}(U)$.

Thus, Ψ is a continuous mapping. So, it is a sheaf morphism.

2. Ψ is a sheaf homomorphism. In fact, $\Psi \mid H_{(x_1,x_2)} = \Psi^*: H_{(x_1,x_2)} \to (H_1)_{x_1} x (H_2)_{x_2}$ is a homomorphism for every $(x_1,x_2) \in X_1xX_2$, since

$$\Psi^* ([\alpha_1].[\alpha_2]) = \Psi^*([\alpha_1.\alpha_2]) = ([p_1o\alpha_1.\alpha_2], [p_2o\alpha_1.\alpha_2]).$$

$$\begin{aligned}\Psi^* ([\alpha_1]).\Psi^*] ([\alpha_2]) &= ([p_1o\alpha_1], [p_2o\alpha_1]). ([p_1o\alpha_2], [p_2o\alpha_2]) \\ &= ([p_1o\alpha_1].[p_1o\alpha_2]), [p_2o\alpha_1]. [p_2o\alpha_2]) \\ &= ([p_1o\alpha_1.p_1o\alpha_2], [p_2o\alpha_1.p_2o\alpha_2]) \\ &= ([p_1o\alpha_1.\alpha_2], [p_2o\alpha_1.\alpha_2]).\end{aligned}$$

3. Ψ is an injection. Because if, $\Psi ([\alpha_1]) = \Psi ([\alpha_2])$ for any $[\alpha_1]$, $[\alpha_2] \in H$, then $([p_1o\alpha_1], [p_2o\alpha_1]) = ([p_1o\alpha_2], [p_2o\alpha_2])$ and $[p_1o\alpha_1] = [p_1o\alpha_2]$, $[p_2o\alpha_1] = [p_2o\alpha_2]$. Therefore, $[p_1o\alpha_1] \overset{F_1}{\sim} [p_1o\alpha_2]$ rel. x_1 and $[p_2o\alpha_1] \overset{F_2}{\sim} [p_2o\alpha_2]$ rel. x_2. Let us now define a homotopy $F: IxJ \to X_1xX_2$ by means of the homotopies F_1 and F_2 as $F (x, t) = (F_1 (x,t), F_2 (x, t))$. Then F is continuous. On the other hand,

$$F(x,0) = (F_1(x,0), F_2(x,0)) = (p_1o\alpha_1, p_2o\alpha_1) = \alpha_1$$

$$F(x,1) = (F_1(x,1), F_2(x,1)) = (p_1o\alpha_2, p_2o\alpha_2) = \alpha_2$$

and

$$F(0,t) = F(1,t) = (x_1,x_2).$$

Thus, $\alpha_1 \overset{F}{\sim} \alpha_2$ rel. (x_1,x_2) and so $[\alpha_1] = [\alpha_2]$.

4. Ψ is a surjection. Because if, β_1 and β_2 are closed arcs at x_1 and x_2 respectively, then $[\beta_1]_{x_1} \in (H_1)_{x_1}$, $[\beta_2]_{x_2} \in (H_2)_{x_2}$ and $([\beta_1]_{x_1}, [\beta_2]_{x_2}) \in (H_1)_{x_1}x(H_2)_{x_2}$. Let us now define a closed arc at (x_1, x_2) as

$$\alpha (x) = \begin{cases} (\beta_1(2x),x_2), & 0 \leq x \leq 1/2 \\ (x_1, \beta_2(2x-1), & 1/2 \leq x \leq 1. \end{cases}$$

Therefore, $[\alpha] \in H_{(x_1,x_2)}$ and $\Psi ([\alpha]) = ([p_1o\alpha], [p_2o\alpha]) = ([\beta_1], [\beta_2])$. So Ψ is onto, since $(x_1,x_2) \in X_1xX_2$ is an arbitrary fixed element.

5. Ψ has an invers mapping $\Psi^{-1} : H_1xH_2 \to H$, since Ψ is a bijection. Ψ^{-1} is continuous, since Ψ is a sheaf morphism.

Thus Ψ is a sheaf isomorphism and $H \cong H_1xH_2$.

Let us now state the following theorem.

Theorem 2.7. Let the pairs (X_1, H_1) and (X_2,H_2) be given. Then the sheaf H and the sehaf $H^* = H_1 \oplus H_2$ are isomorphic.

ÖZET

X lokal eğrisel irtibatlı bir topolojik uzay olsun. [1] deki çalışmamızda X üzerinde Esas Grupların Demetini inşa edip bazı karakterizasyonlar vermiştik. [2] deki çalışmamızda ise [1] de verilen karakterizasyonların karşıtları verilmiş ve kesitlerin grubu ile ilgili bazı neticeler elde edilmişti. Bu çalışmamızda, "Genelleştirilmiş Whitney Toplamı" tarifi verilerek $H^* = H_1 \oplus H_2$ demeti inşa edilmiş ve gösterilmiştir ki bu demet $H_1 x H_2$ demetine izomorftur. Burada H_1 ve H_2 sırasıyla X_1 ve X_2 de [1] deki yöntemle inşa edilen demetlerdir. Daha sonra, gösterdik ki, (X_1, H_1), (X_2, H_2) herhangi iki çift olmak üzere, H^* demeti, $X_1 x X_2$ üzerinde [1] deki yöntemle inşa edilen H demetine izomorftur.

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