COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES DE L'UNIVERSITÉ D'ANKARA

Série A_1 Mathematiques

TOME : 33 ANNÉE : 1984

Banach Limits And Infinite Matrices (II)

by

SHANTI LATA MISHRA

29

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences
de l'Universite d'Ankara

La Revue "Communications de la Faculté des Sciences de l'Université d'Ankara" est un organe de publication englobant toutes les diciplines scientifique représentées à la Faculté des Sciences de l'Université d'Ankara.

La Revue, jusqu'à 1975 àl'exception des tomes I, II, III etait composé de trois séries

Série A: Mathématiques, Physique et Astronomie,
Série B: Chimie,
Série C: Sciences Naturelles.

A partir de 1975 la Revue comprend sept séries:

Série A_1: Mathématiques,
Série A_2: Physique,
Série A_3: Astronomie,
Série B: Chimie,
Série C_1: Géologie,
Série C_2: Botanique,
Série C_3: Zoologie.

A partir de 1983 les séries de C_2 Botanique et C_3 Zoologie on été réunies sous la seule série Biologie C et les numéros de Tome commencerons par le numéro 1.

En principle, la Revue est réservée aux mémories originaux des membres de la Faculté des Sciences de l'Université d'Ankara. Elle accepte cependant, dans la mesure de la place disponible les communications des auteurs étrangers. Les langues Allemande, Anglaise et Française serint acceptées indifféremment. Tout article doit être accompagnés d'un resume.

Les article soumis pour publications doivent être remis en trois exemplaires dactylographiés et ne pas dépasser 25 pages des Communications, les dessins et figuers portes sur les feulles séparées devant pouvoir être reproduits sans modifications.

Les auteurs reçoivent 25 extrais sans couverture.

l'Adresse : Dergi Yayın Sekreteri,
Ankara Üniversitesi,
Fen Fakültesi,
Beşevler—Ankara
TURQUIE

Banach Limits And Infinite Matrices (II)

SHANTI LATA MISHRA

Department of Mathematics, Sambalpur University,

Jyoti Vihar, 768019 Sambalpur, Orissa, India.

(Received March 2, 1984; Revised October 8, 1984 and accepted December 28, 1984)

ABSTRACT

An inequality sharper than that of Knopp's Core inequality was proved in [3]. In the present paper a generalised result of the above inequality for row finite matrices is proved with the help of a sublinear functional Ω_B Some sets which arise in connection with Ω_B are also characterised.

INTRODUCTION

Let m denote the Banach space of all *bounded* real sequences $x = \{x_n\}_{n=1}^{\infty}$, normed by $\|x\| = \sup_n |x_n|$. We write

$$m_0 = \{x \in m: \sup_n \left| \sum_{i=1}^{n} x_i \right| < \infty\}.$$

Define $L: m \to R$ by $L(x) = \lim_n \sup x_n$. The space c of all *convergent* real sequences is a closed subspace of m.

Banach limits [1] are linear functionals G on the space m satisfying conditions:

(i) $x \geq 0 \Rightarrow G(x) \geq 0$,

(ii) $G(e) = 1$,

(iii) $G(\sigma x) = G(x)$,

where $e = (1,1 \ldots)$ and $\sigma: m \to m$ is defined by $(\sigma x_n) = x_{n+1}$. Condition (iii) is the same thing as saying that G is σ-invariant on m and σ is called a *shift operator*. Let β denote the set of all Banach limits on m.

If P is a sublinear functional on m, we write $\{m,P\}$ to denote the set of all linear functionals Q on m such that $Q \leq P$ (that is, $Q(x) \leq P(x)$ for all $x \in m$). A sublinear functional P on m *generates* Banach limits if for a linear functional G on m, $G \leq P \Rightarrow G \in \beta$; (that is, if $\{m,P\} \subset \beta$). A sublinear functional P *dominates* Banach limits if $G \in \beta \Rightarrow G \leq P$; (that is if $\beta \subset \{m,P\}$). Thus if P both dominates and generates Banach limits then $\beta = \{m,P\}$.

Let $A = (a_{nk})$ be an infinite matrix of real numbers and write

$A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ if it converges for all $n > 0$. We then write $Ax = \{A_n(x)\}_{n=1}^{\infty}$. Note that the matrix A is called *regular* if $A: c \rightarrow c$ and $\lim Ax = \lim x$. The Silverman-Toeplitz conditions for a regular matrix are the following:

(i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$,

(ii) $\lim_n a_{nk} = 0$, for fixed k,

(iii) $\lim_n \sum_k a_{nk} = 1$.

A matrix A is called *strongly regular* [4] if it is regular and

$$\lim_n \sum_k |a_{nk} - a_{n,k+1}| = 0.$$

We say that $A = (a_{nk})$ is *almost positive* if $\lim_n \sum_k a^-_{nk} = 0$ (if $\lambda \in R$, λ^+ means $\max(\lambda, 0)$ and λ^- means $\max(-\lambda, 0)$). If A is regular, it is almost positive if and only if $\lim_n \sum_k |a_{nk}| = 1$ (see [7]).

The main object of this paper is to establish an inequality for a row finite matrix A and for a sublinear functional defined on m_B for a normal matrix B. This is proved in section 3, and it is a generalisation of Theorem 3 of [3] for a row finite matrix. In section 4 sets which arise in connection with Ω_B have been studied. Section 5 deals with a set of section 4 where m_0 is replaced by a bounded subspace V of m.

2. Let s be the set of all real sequences $x = \{X_n\}_{n=1}^{\infty}$. We write $m_A = \{x \in s: Ax \in m\}$, $m_{A0} = \{x \in s: Ax \in m_0\}$. It is evident that m_A

is a linear space and m_{A0} is a subspace, Further if we define, for $x \in m_A$, $\|x\| = \sup_n |\sum_k a_{nk}x_k|$ then it is a seminorm on m_A. It is a norm if A is invertible. It is also familiar that

$$A: m \to s \Leftrightarrow \sum_k |a_{nk}| < \infty \text{ (for each n);}$$

$$A: m \to m \Leftrightarrow \|A\| = \sup_n \sum_k |a_{nk}| < \infty.$$

Let c_A be the summability field of A; that is,

$$c_A = \{x \in s: L(Ax) = - L(-Ax)\}.$$

It is evident that

$$c_A \subset m_A \tag{2.1}$$

It is also easily seen that

$$m \cap m_A = m \Leftrightarrow \|A\| < \infty. \tag{2.2}$$

It is in order to quote the following theorem.

Theorem: (Mazur-Orlicz [6]). Let A be a regular matrix. Then $c_A \cap c' \neq \varnothing \Rightarrow c_A \cap m' \neq \varnothing$; where c' and m' are the complementary sets of c and m respectively. In otherwords, if a regular matrix evaluate some divergent sequence, then it must evaluate an unbounded sequence; that is, if a regular matrix evaluates no unbounded sequence, then it evaluates only convergent sequences. From (2.2) we have $\|A\| < \infty \Rightarrow m \subset m_A$ and there are important cases where m is a proper subset of m_A. For example, if A is regular such that A evaluates some divergent sequence (infact, these cases are only important), then the above theorem gives that $c_A \cap m' \neq \varnothing$ and therefore from (2.1) we have $m_A \cap m' \neq \varnothing$.

Let $w: m \to R$ be defined by

$$w(x) = \inf_{z \in m_0} L(x+z)$$

It is easy to see that w is a sub-linear functional. The result which was proved in [3] is the following:

Theorem: $\limsup_n A_n(x) \leq w(x)$ $(x \in m)$

if and only if the matrix A is almost positive an strongly regular.

The following lemmas are required to prove the main Theorem and the proposition.

Lemma 1: (Knopp's Core Theorem) $L(Ax) \leq L(x)$ $(x \in m)$ if and only if A is almost positive and regular.

Lemma 2: (Simons [7], Corollary 12, Theorem 11). If

(i) $\sum_k |a_{nk}| < \infty$ (for each n)

(ii) $a_{nk} \to 0$ $(n \to \infty)$ for fixed k,

then there exists $y \in m$: $\|y\| \leq 1$ and

$$\limsup_n \sum_k a_{nk} y_k = \limsup_n \sum_k |a_{nk}|.$$

3. Now suppose that $\|B\| < \infty$ and we write, for any real matrix B, and for $x \in m_B$

$$\Omega_B(x) = \inf_{z \in m_{Bo}} L(B(x+z)). \tag{3.1}$$

The function Ω_B: $m_B \to R$ is well-defined if we suppose that

$$\limsup_n B_n z \geq 0, \quad (z \in m_{Bo}), \tag{3.2}$$

(see Devi [3], regarding the functional q_y before the statement of Theorem 1).

In the case

$$b_{nk} \to 0 \quad (n \to \infty, \ k \text{ fixed})$$

by Abel's transformation,

$$B_n z = \sum_k (b_{nk} - b_{n,k+1}) y_k, \quad y \in m \text{ and } z \in m_0$$

where $y = \{y_n\} = \{ \sum_{v=0}^{n} z_v \}$.

Further if

$$\lim_n \sum_k |b_{nk} - b_{n,k+1}| = 0 \tag{3.3}$$

then, since, for $y \in m$ (that is, $z \in m_0$)

$$|B_n z| \leq \|y\| \sum_k |b_{nk} - b_{n,k+1}|,$$

it follows that, in the case (3.3) holds, $\lim_n B_n z = 0$ $(z \in m_0)$. Hence in the case $m_{B0} \subset m_0$ the requirement (3.2) is fulfilled. Now I am in a position to state the first theorem:

Theorem 1: *Let B be a normal matrix such that condition* (3.2) *holds. Then for a row finite matrix A*

$$L(Ax) \leq \Omega_B(x) \quad (x \in m_B) \tag{3.4}$$

if and only if AB^{-1} is almost positive and strongly regular.

Remark: By taking B = I (identity matrix) we obtain Theorem 3 of [3] for a row finite matrix.

For the proof of Theorem 1, I need to prove the following proposition which gives a theorem similar to the Knopp's Core theorem in the case B = I and A, a row finite matrix.

Proposition 1: *Let B be a normal matrix. Then for a row finite matrix A,*

$$\limsup_n A_n(x) \leq \limsup_n B_n(x) \text{ for all } x \in m_B \tag{3.5}$$

if and only if AB^{-1} is regular and almost positive.

Proof: (Sufficiency) Since B is a normal matrix (see [5]), it is row finite and B^{-1} is also row finite. Let $C = AB^{-1}$. Since $CBx = (AB^{-1}) Bx = A(B^{-1}B)x = Ax$, it follows that

$$L(Ax) = L(CBx) \tag{3.6}$$

The associative property of infinite matrices A, B^{-1} and B is justified for row finite matrices (see Cooke [2]). By the sufficiency part of Lemma 1,

$$L(Cy) \leq L(y) \text{ for all } y \in m.$$

Since for all $x \in m_B$, $Bx \in m$, we have from the above inequality that

$$L(CBx) \leq L(Bx).$$

As $L(Ax) = L(CBx)$ by (3.6), we have proved the sufficiency.

Necessity: $-L(-Bx) \leq -L(-Ax) \leq L(Ax) \leq L(Bx)$, $(x \in m_B)$. Hence it follows that

$$L(Bx) = -L(-Bx) \Rightarrow L(Ax) = -L(-Ax),$$

that is,

$$\{x: Bx \in c\} \subset \{x: Ax \in c\}$$

and

$$\lim_n B_n x = \lim_n A_n x. \tag{3.7}$$

If $y \in c$, then $y \in m$. As B is normal there is an $x \in s$ such that $Bx = y$ or $x = B^{-1} y$. Now by using (3.7) we have,

$\lim y_n = \lim B_n x = \lim A_n x = \lim A_n(B^{-1}y) = \lim (AB^{-1})_n y = \lim C_n y$
Hence $C = AB^{-1}$ is a regular matrix.

Now since C is regular, the requirement of lemma 2 is satisfied. Hence there exists $y \in m$: $\|y\| \leq 1$ and

$$L(Cy) = \lim_n \sup \sum_k |c_{nk}| \tag{3.8}$$

Now given y as above, define x by $x = B^{-1} y$ so that $\|Bx\| \leq 1$. Since $L(Bx) \leq 1$, using (3.5) and (3.6) we get

$$L(Ax) = L(Cy) \leq 1. \tag{3.9}$$

Now it follows from (3.8) and (3.9) that

$$\lim_n \sup \sum_k |c_{nk}| \leq 1. \tag{3.10}$$

But since

$$\lim_n \sup \sum_k |c_{nk}| \geq \lim_n \sum_k c_{nk} = 1$$

it follows from (3.10) that

$$\lim_n \sum_k |c_{nk}| = 1.$$

Hence C is almost positive.

This completes the proof of the proposition.

Proof of the Theorem 1: (Sufficiency) Since $C = AB^{-1}$ is almost positive and regular, it follows, by proposition 1, that

$$L(A(x+z)) \leq L(B(x+z)) \quad (x \in m_B,\ z \in m_{Bo}).$$

Now taking the infimum with respect to $z \in m_{B_0}$ in the above inequality, we have

$$\Omega_A(x) \leq \Omega_B(x). \tag{3.11}$$

Since L(Ax) is sublinear, it follows that

$$\Omega_A(x) \geq \inf_{z \in m_{B_0}} |L(Ax) - L(-Az)|. \tag{3.12}$$

But for $z \in m_{B_0}$, $Az = CBz = Dy$

where

$$D = (d_{nk}) = (c_{nk} - c_{n,k+1}),$$

$$y = \{y_n\} = \{ \sum_{v=0}^{n} B_v(z) \} \in m. \tag{3.13}$$

Since C is strongly regular and $y \in m$, it follows that

$$L(Az) = L(Dy) = 0.$$

Hence it follows from (3.12) that

$$\Omega_A(x) \geq \inf_{z \in m_{B_0}} L(Ax) = L(Ax). \tag{3.14}$$

Now the sufficiency follows from (3.11) and (3.14).

Necessity: Suppose that (3.4) holds, since trivially $\Omega_B(x) \leq L(Bx)$, $(x \in m_B)$ it follows from (3.4) that $L(Ax) \leq L(Bx)$ $(x \in m_B)$. Hence by proposition 1, $C = AB^{-1}$ is almost positive and regular.

Since we know (see Devi [3], Theorem 1 (i))

$\Omega_B(x) = 0(x \in m_{B_0})$, it follows from (3.4) that

$$L(Ax) \leq 0 \ (x \in m_{B_0});$$

that is,

$$L(CBx) \leq 0 \ (x \in m_{B_0});$$

that is,

$$L(Dy) \leq 0 \ (y \in m), \tag{3.15}$$

where D and y are given by (3.13).

Now since the matrix D satisfies the conditions of Lemma 2 (as C is regular) there exists $y_0 \in m$: $\|y_0\| \leq 1$ and

$$L(Dy_0) = \lim_n \sup \sum_k |d_{nk}| \geq 0 \tag{3.16}$$

Now define x_0 by

$$x_0 = B^{-1} (\sigma y_0 - y_0),$$

so that

$$\sigma y_0 - y_0 = Bx_0.$$

Hence $y_0 \in m \Leftrightarrow Bx_0 \in m_0 \Leftrightarrow x_0 \in m_{B_0}$.

Now taking y to be y_0 in (3.15) together with relation (3.16), we have

$$\lim_n \sum_k |d_{nk}| = \lim_n \sum_k |c_{nk} - c_{n,k+1}| = 0.$$

Hence C is strongly regular.

This completes the proof.

Corollary 1: *Let the conditions of Theorem* 1 *hold. Then*

$$L(Ax) \leq \Omega_A(x) \leq \Omega_B(x) \leq L(Bx).$$

Proof: First inequality follows from (3.14), second inequality from (3.11) and the last one is trivial.

4. It is easy to see that

$$\Omega_A(x) \leq L(Ax) \; (x \in m) \tag{4.1}$$

but by Theorem 3 of Devi [3], we have

$$L(Ax) \leq w(x) \; (x \in m) \tag{4.2}$$

if and only if A is almost positive and strongly regular. Hence combining (4.1) and (4.2) we have

Theorem 2: *Let A be almost positive and strongly regular. Then*

$$\Omega_A(x) \leq w(x) \; (x \in m). \tag{4.3}$$

In other words,

$$\{m, \Omega_A\} \subset \beta,$$

that is, Ω_A generates Banach limits. This is justified as

$\beta = \{m,w\}$ (see Theorem 1′(i) [3]).

It is clear from (4.3) that

$$-w(-x) \leq -\Omega_A(-x) \leq \Omega_A(x) \leq w(x) \quad (x \in m)$$

Since $w(x) = -w(-x)$ implies that $-\Omega_A(-x) = \Omega_A(x)$, it follows that

$$\{x \in m: w(x) = -w(-x)\} \subset \{x \in m: \Omega_A(x) = -\Omega_A(-x)\}.$$

that is,

$$\hat{c} \subset S_1,$$

if A is almost positive and strongly regular, where

$\hat{c} = \{x \in m: x \text{ has unique Banach limit}\}$

$= \{x \in m: w(x) = -w(-x)\}$,

$$= \left\{ x \in m: \frac{x_n + x_{n+1} + \ldots + x_{n+p}}{p+1} \to \text{a limit as } p \to \infty, \text{ uniformly in } n \right\},$$

and

$$S_1 = \{x \in m: \Omega_A(x) = -\Omega_A(-x)\}.$$

$\hat{c}$ is called the set of all almost convergent sequences (see Lorentz [4]). In what follows, we want to examine if the set S_1 can have a simpler characterisation like the set $\hat{c}$.

Write

$$S_0 = \{x \in m: \sum_k a_{nk}(x_k + z_k) \text{ converges uniformly in } z \in m_0\}$$

$$S_2 = \{x \in m: \sum_k a_{nk}(x_k + z_k) \text{ converges for all } z \in m_0\}.$$

We now prove

Theorem 3:

(i) $S_0 \subset S_1$

(ii) $S_1 \subset S_2$ if $\sum_k |a_{nk} - a_{n,k+1}| \to 0$ as $n \to \infty$.

Proof: Given $x \in S_0$ and $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon)$:

$$s_1 - \varepsilon < \sum_k a_{nk}(x_k + z_k) < s_1 + \varepsilon \tag{4.4}$$

for all $z \in m_0$ and for all $n \geq n_0$, where

$$s_1 = \lim_n \sum_k a_{nk} (x_k + z_k),$$

and s_1 is independent of $z \in m_0$. Taking lim sup over n and then the infimum over z in (4.4) we obtain:

$$s_1 - \varepsilon \leq - \Omega_A(-x) \leq \Omega_A(x) \leq s_1 + \varepsilon \tag{4.5}$$

Since ε is arbitrary, we get the first inclusion relation.

Next suppose that $x \in S_1$ and $\Omega_A(x) = - \Omega_A(-x) = s_1$. From $\Omega_A(x) = s_1$, we obtain, given $\varepsilon > 0$ there exists $z' \in m_0$ and $n_1 \in N$:

$$A_n(x + z') = \sum_k a_{nk} (x_k + z'_k) < s_1 + \varepsilon, \tag{4.6}$$

for all $n \geq n_1$. Now for $z \in m_0$,

$$A_n(x + z) = A_n(x + z') + A_n(z - z'). \tag{4.7}$$

Since $\sum_k |a_{nk} - a_{n,k+1}| \to 0$ as $n \to \infty$, we obtain

$$A_n(z - z') \to 0 \text{ as } n \to \infty,$$

that is, given $\varepsilon > 0$, there exists $n_2 \in N$:

$$A_n(z - z') < \varepsilon \ (n \geq n_2). \tag{4.8}$$

Now from (4.6), (4.7) and (4.8) we have

$$A_n(x + z) < s_1 + 2\varepsilon \text{ for all } n > n_3 = \max(n_1, n_2).$$

Similarly we have

$$A_n(x + z) > s_1 - 2\varepsilon \text{ for all } n \geq n_4 \in N,$$

so that

$$|A_n(x + z) - s_1| < 2\varepsilon \text{ for all } n \geq n_5 = \max(n_3, n_4).$$

Hence $x \in S_2$ and this proves the second inclusion relation.

5. The set S_0 defined in Section 4 is usually empty. In fact it is non-empty only if the matrix A has a finite number of non-zero rows. In view of this it is evident that the inclusion (i) of Theorem 3 is proper because when A is almost positive and strongly regular then $\hat{c} \subset S_1$ (see below Theorem 2).

Now the natural question arises as to what sublinear functional Ψ will generate the set S_0 in the sense that

$$S_0 = \{x \in m: \Psi(x) = -\Psi(-x)\}.$$

Towards this end, we define $\Psi_A: m \to R$ by

$$\Psi_A(x) = \limsup_n \sup_{z \in V} \sum_k a_{nk}(x_k + z_k)$$

where V is a subspace of m.

Since

$$\Psi_A(x) \geq \limsup_n \sum_k a_{nk} x_k$$

and if $x \in m$, $\|A\| < \infty$ then Ψ_A is bounded from below. Ψ_A is also bounded from above if V is a bounded subspace. In this case Ψ_A is well-defined. Now we have the following

Theorem 4: Let V be a bounded subspace of m and let $\|A\| < \infty$. *Write*

$$\hat{S}_0 = \{x \in m: \sum_k a_{nk}(x_k+z_k) \to \text{a limit as } n\to\infty \text{ uniformly in } z \in V\}.$$

Then

$$\hat{S}_0 = \{x \in m: \Psi_A(x) = -\Psi_A(-x)\}.$$

Proof: Suppose that $\sum_k a_{nk}(x_k + z_k) \to \alpha$ uniformly in $z \in V$.

Then given $\varepsilon > 0$, there exists $n_0 \in N$:

$$\alpha - \varepsilon < \sum_k a_{nk}(x_k + z_k) < \alpha + \varepsilon \text{ for all } z \in V,\ n \geq n_0.$$

Now taking sup. with respect to z and then lim sup. with respect to n, we have

$$\alpha - \varepsilon \leq -\Psi_A(-x) \leq \Psi_A(x) \leq \alpha + \varepsilon.$$

Since ε is arbitrary, we obtain

$$\Psi_A(x) = -\Psi_A(-x) = \alpha.$$

Conversely suppose that $\Psi_A(x) = \alpha = -\Psi_A(-x)$. Then we shall have

$$\alpha - \varepsilon < \sum_k a_{nk}(x_k + z_k) < \alpha + \varepsilon$$

foll all $z \in V$, $n \geq n_0$, from which follows that

$$\sum_k a_{nk}(x_k + z_k) \to \alpha \text{ uniformly in } z \in V.$$

This completes the proof.

REFERENCES

1- **S. Banach,** *Theorie des operations lineaires* (Warszawa, 1932).

2- **R.G. Cooke,** *Infinite matrices and sequences spaces.* (Macmillan, 1950).

3- **S.L. Devi,** Banach limits and Infinite matrices, *Jour. Lond. Math. Soc.* (2), 12 (1976) 397–401.

4- **G.G. Lorentz,** A contribution to the theory of divergent sequences, *Acta Math.*, 80 (1948), 167–190.

5- **I.J. Maddox,** *Elements of Functional Analysis.* (Cambridge University Press 1970).

6- **S. Mazur, and W. Orlicz,** On linear methods of summability, Statement in comptes Rendues, 196 (1933) 32–34, Full account in *Studia Math.* 14 (1955), 129–160.

7- **S. Simons,** Banach Limits, Infinite matrices and sublinear functionals, *Jour. Math. Anal. and Appl.* (3), 26, (1969) 640–650.