COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1 : Mathematique

TOME : 33 ANNÉE : 1984

On The Absolute Summability Factors Of Infinite Series

by

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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

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TURQUIE

On The Absolute Summability Factors Of Infinite Series

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(Received April 19, 1984 and accepted August 13, 1984)

ABSTRACT

In this paper, a theorem on $|\bar{N},p_n|_k$ summability factors, which generalizes the result of Khan [2], has been proved.

1. Let Σa_n be a given infinite series, with the sequence of partial sums (s_n) and let (p_n) be a sequence of positive real constants such that

$$P_n = p_0 + p_1 + p_2 + \dots + p_n \to \infty, \text{ as } n \to \infty, (P_{-1} = p_{-1} = 0).$$

We write

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v, \quad u_n = \frac{1}{P_n} \sum_{v=1}^{n} P_{v-1} a_v.$$

The series Σa_n is said to be summable $|\bar{N}, p_n|_k$, $(k \geq 1)$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty \ ([1]).$$

In the special case when $p_n = 1$, for all values of n (resp. $k = 1$), $|\bar{N},p_n|_k$ summability is the same as $|C,1|_k$ (resp. $|\bar{N},p_n|$) summability.

We write throughout, for any sequence (r_n),

$$\Delta r_n = r_n - r_{n+1}, \ \Delta^2 r_n = \Delta (\Delta r_n).$$

We observe that whenever $k = 1$, our theorem includes the following theorem of Khan [2].

Theorem A. Let $p_n > 0$ $(n = 1,2,3,...)$ and $(n+1)\, p_n = O\,(P_n)$. If

$$\sum_{v=1}^{n} (p_v / P_{v-1})\ |u_v| = O\,(\gamma_n)$$

where (γ_n) is a positive monotonic non-decreasing sequence, and if the sequences (λ_n) and (γ_n) are such that

(i) (a) $\lambda_n \gamma_n = O\,(1)$; (i) (b) $p_n \triangle \gamma_n = O\,(|\triangle p_n|\, \gamma_n)$ as $n \to \infty$,

(ii) $\sum_{n=1}^{\infty} P_n\ |\triangle (1/p_n)|\ \gamma_n\ |\triangle \lambda_{n+1}| < \infty$,

(ii) $\sum_{n=1}^{\infty} (P_n / p_n)\ \gamma_n\ |\triangle^2 \lambda_n| < \infty$,

then the series $\Sigma\, a_n \lambda_n$ is summable $|\bar{N}, p_n|$.

2. We shall now prove the following theorem.

Theorem. Let $p_n > 0$ $(n = 1,2,3,...)$ and $(n+1)\, p_n = O\,(P_n)$. If

$$\sum_{v=1}^{n} (p_v / P_{v-1})\ |u_v|^k = o\,(\gamma_n),$$

where (γ_n) is a positive monontonic on-decreasing sequence, and if the sequences (λ_n) and (γ_n) are such that

(i) (a) $\lambda_n \gamma_n = O\,(1)$,

(i) (b) $p_n \triangle \gamma_n = O\,(|\triangle p_n|\, \gamma_n)$,

(i) (c) $(P_n\, |\triangle \lambda_n|)^{k-1} = O\,(p_n)^{k-1}$,

(i) (d) $(P_n\, |\lambda_n|)^{k-1} = O\,(P_{n-1})^{k-1}$ as $n \to \infty$,

(ii) $\sum_{n=1}^{\infty} P_n\ |\triangle (1/p_n)|\ \gamma_n\ |\triangle \lambda_{n+1}| < \infty$,

(ii) $\sum_{n=1}^{\infty} (P_n / p_n)\ \gamma_n\ |\triangle^2 \lambda_n| < \infty$,

then the series $\Sigma\, a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $(k \geqq 1)$.

3. We require the following lemmas for the proof of our theorem.

Lemma 1 [2]. Let $p_n > 0$, for all $n \geqq 0$, such that $(n+1)\, p_n = 0(P_n)$. If $\lambda_n \gamma_n = 0\ (1)$ and

$$\sum_{n=1}^{\infty} (P_n/p_n)\, \gamma_n\, |\triangle^2 \lambda_n| < \infty,$$

where (γ_n) is a positive monotonic non-decreasing sequence, then

$$\sum_{n=1}^{\infty} \gamma_n\, |\triangle\lambda_n| < \infty.$$

Lemma 2 [2]. If the sequences (λ_n) and (γ_n) satisfy the same conditions as in the Theorem A, then

$$(P_{n-1}/p_n)\, \gamma_n\, |\triangle\lambda_n| = 0\ (1), \text{ as } n \to \infty.$$

4. **Proof Of The Theorem.** Let T_n denote the $(\overline{N}, p_n)$ mean of the series $\Sigma\, a_n\lambda_n$ and

$$u_n = \frac{1}{P_n} \sum_{v=1}^{n} P_{v-1}\, a_v.$$

Then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1}\, a_v \lambda_v.$$

Using Abel's transformation, we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v u_v \triangle \lambda_v$$

$$+ \frac{p_n u_n \lambda_n}{P_{n-1}} = T_{n,1} + T_{n,2}, \text{ say.}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\, T_{n,r}\, |^k < \infty, \text{ for } r = 1,2.$$

Now, applying Hölder's inequality, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,1}|^k \leq \sum_{n=2}^{m+1} \frac{p_n}{p_n\ (P_{n-1})^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v\ |\triangle \lambda_v|\ |u_v| \right\}^k$$

$$\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n\ P_{n-1}} \sum_{v=1}^{n-1} (P_v/p_v)^k\ p_v\ |\triangle\lambda_v|^k\ |u_v|^k \ x \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^k\ p_v\ |\triangle\lambda_v|^k\ |u_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n\ P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} (P_v|\triangle\lambda_v|/p_v)^{k-1}\ |\triangle\lambda_v|\ |u_v|^k = 0\ (1) \sum_{v=1}^{m} |\triangle\lambda_v|\ |u_v|^k$$

$$= O(1) \sum_{v=1}^{m} (P_{v-1}/p_v)\ |\triangle\lambda_v|\ (p_v/P_{v-1})\ |u_v|^k = O(1) \sum_{v=1}^{m-1} (P_{v-1}/p_v)\ |\triangle^2\lambda_v|\gamma_v$$

$$+ O(1) \sum_{v=1}^{m-1} |\triangle\lambda_{v+1}|\ \gamma_v + O(1) \sum_{v=1}^{m-1} P_v\ |\triangle (1/p_v)|\ |\triangle\lambda_{v+1}|\ \gamma_v$$

$$+ O(1)\ (P_{m-1}/p_m)\ \gamma_m\ |\triangle\lambda_m| = O(1), \text{ as } m \to \infty$$

by hypotheses and Lemma 1 and 2.

Finally, we have

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,2}|^k = \sum_{n=1}^{m} (P_n\ |\lambda_n|/P_{n-1})^{k-1}\ (p_n/P_{n-1})\ |\lambda_n|\ |u_n|^k$$

$$= O(1) \sum_{n=1}^{m} (p_n/P_{n-1})\ |\lambda_n|\ |u_n|^k = O(1) \sum_{n=1}^{m-1} \gamma_n\ |\triangle\lambda_n|$$

$$+ O(1)\ |\lambda_m|\ \gamma_m = O(1),\ m \to \infty$$

by hypotheses and Lemma 1.

This completes the proof of the theorem.

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ÖZET

Bu çalışmada Khan [2]'ın bir sonucunu genelleştiren $|\overline{N},p_n|_k$ toplanabilme çarpanıyla ilgili bir teorem ispat edilmiştir.