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A GENERALIZATION OF THE GAUSS-BONNET THEOREM

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ABSTRACT

We find a new expression of the Gauss-Bonnet Theorem [Haschke (1948), Chern (1944), Hichs (1974)] for the volume of the hyperspherical regions which are surrounded by the point orbits of the hyperspherical motions [Hac salihoğlu (1977)], by using the Steiner's vector [Flanders (1966)]. The new expression satisfiess the well-known Bonnet area formulae [Blaschke (1948), Hacısalihoğlu (1972)] and the Gauss-Bonnet Theorem for even dimensional manifolds [Chern (1944)]. At the and we obtain a genaralization for the Holditch's Theorem [Hacısalihoğlu (1971)] on hyperspheres.

We find a new expression of the Gauss-Bonnet Theorem for the volume of the hyperspherical regions which are surrounded by the point orbits of the hyperspherical motions, by using the Steiner's vector. The new expression satisfiess the well-known Gauss-Bonnet area formulae and the Gauss-Bonnet Theorem for even dimensional manifolds. At the and we obtain a generalization for the Holditch's Theorem on hyperspheres.

GENERALIZATION OF THE GAUSS-BONNET THEOREM TO THE CASE n > 3.

In this section, we express and prove the Gauss-Bonnet Theorem for (n—1)-dimensional hypersphere S^{n-1} in n-dimensional Euclidean space E^n by using the Steiner vector $\vec{V}$, which is defined on a closed region A on S^{n-1} as in the following

DEFINITION 1: Let A be a compact region on the hypersphere S^{n-1} and X is an arbitrary fixed point of S^{n-1}. Then the Steiner vector defined as

$$\vec{V} = \int_A \vec{X}\, V_S$$

where V_S is the volume element of S^{n-1}.

The definition on the Steiner vector in euclidean 3-space E^3 is given in [Hacısalihoğlu (1971)] by the following formula

$$\vec{V} = \oint w \, \vec{p}$$

where w is pfafian. The Steiner point S_x defined as

$$S_x = \frac{\oint w \, X}{\oint w}$$

in [Hacısalihoğlu (1971)]. Notice that the Steiner vector is the numerator of this quotient. For hypersurfaces, the Steiner point S(M) is defined as

$$S(M) = \frac{\int_M x \, K \, d\sigma}{\int_M K \, d\sigma}$$

in [Fladers (1966)], where K is the Gaussian curvature of the hypersurface M and $d\sigma$ is the volume element of M.

When the M is a sphere S^{n-1} then K=1. Thus, we have

$$S(S^{n-1}) = \frac{\int_{S^{n-1}} X \, d\sigma}{\int_{S^{n-1}} d\sigma}$$

For the relation between the Steiner point and the Steiner vector in Euclidean 3-space, we introduce the definition of the Steiner vector on the hypersphere in n-Euclidean space E^n as in Definition 1. Thus, Definition 1 can be taken as a generalization of the Steiner vector in 3–dimensional space E^3.

THEOREM 1: Let (X) be the orbit, on K′, of an arbitrary fixed point X of K. The spherical volume F_x bounded by the closed region A may be calculated from

$$F_x = \langle \vec{V}, \vec{X} \rangle \tag{1}$$

where the vector $\vec{V}$ is the Steiner vector belonging to the closed region A.

PROOF: Let $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be a coordinate system such that $\vec{e}_n = \vec{X}$. By the definition of the Steiner vector, we have

$$\vec{V} = \int_A \vec{e}_n \, V_s$$

$$= \vec{e}_n \int_A V_s$$

$$= (0,0,\ldots,0, \int_A V_s).$$

It is seen that the components of $\vec{V}$ with respect to this coordinate system are $V_1 = V_2 = \ldots = V_{n-1} = 0$ and $V_n = \int_A V_s$. From that, we can obtain the following equation

$$V_n = < \vec{V} , \vec{e}_n >$$

or

$$\int_A V_s = < \vec{V} , \vec{e}_n > \tag{2}$$

In this equation, if we write the inner product of the vectors $\vec{V} = (V_1,\ldots,V_n)$ and $\vec{X} = (0,\ldots,0,1)$ instead of V_n, that is to say, $V_n = < \vec{V} , \vec{X} >$, we obtain

$$\int_A V_s = < \vec{V} , \vec{X} > .$$

Since the inner product is independent of the coordinate transforms, we have

$$\int_A V_s = < \vec{V} , \vec{X} >$$

by putting the value of V_n in the formula (2), or

$$F_x = \int_A V_s$$

$$= < \vec{V} , \vec{X} >$$

where F_x and V_s are the volume and volume element of the region A, respectively.

SPECIAL CASE

a) The case n=3:

In this case, the region A is a spherical area that is surrounded by the the curve (X). If we take the total rotation number of the tangential $d\vec{X}$ of the curve (X) with respect to the fixed sphere K′ as 1 and choose the compatible orientation on the sphere, then the relation obtained in the Theorem 1 is the same as the Gauss-Bonnet formula for the real 2-sphere in [Blaschkel (1944)] and the formula (2.5) for the dual 2-sphere in [Hacısalihoğlu (1972)].

b) The case $n > 3$ for n even

For the Gauss-Bonnet Theorem for an even dimensional hypersurface M in $\mathbb{R}^{n+1}$, [Hicks (1974)] gives the following formula

$$\int_M w_{1n+1} \wedge \ldots \wedge w_{nn+1} = \int_M \eta^* (V_S) .$$

The above equation, for a compact region A on the hypersurface M, becomes

$$\int_A w_{1n+1} \wedge \ldots \wedge w_{nn+1} = \int_A V_S .$$

If we take the region A as the region that is surrounded by the closed curve (X) which is the orbit on the unit hypersphere K′ drawn by a fixed point X of the unit hypersphere K, then the volume F_X of the region A has the following expression

$$F_X = \int_A V_S .$$

Since the inner product is independent of the coordinate transforms, as in the proof of Theorem 1, one can show that

$$\int_A V_S = < \vec{V} , \vec{X} > .$$

Thus, we again obtain that

$$F_X = < \vec{V} , \vec{X} > .$$

All the special cases a and b imply that "for all n, even if $n > 3$ n is odd or even number, Theorem 1 is a generalization of the Gauss-Bonnet Theorem".

APPLICATIONS OF THE GAUSS-BONNET TOEOREM AND A GENERALIZATION OF THE HOLDITCH'S THEOREM

Let K and K′ be cocentered unit hyperspheres in Euclidean n-space E^n. We denote the one parameter motion of K with respect to K′ by K/K′. In the following part of this section, the term sphere means that the (n—1)-dimensi onal hypersphere S^{n-1} in Euclidean n-space E^n. During the one parameter closed motion K/K′, two fixed points M, N of K generally plot two closed curves on the fixed sphere K′. Let these two curves encircle the spherical volumes F_M and F_N respectively. Consider another fixed point X of K on the arc $\widehat{MN}$ of a great circle on K of given lenght. During the same motion the point X also draws another closed curve (X) on the sphere K′. Denote by F_X the volume surrounded by (X). Therefore, according to (1)

$$
\begin{aligned}
F_M &= < \vec{M} \,,\ \vec{V} > \\
F_N &= < \vec{N} \,,\ \vec{V} > \\
F_X &= < \vec{X},\ \vec{V} >
\end{aligned} \tag{3}
$$

where the vectors $\vec{M}$, $\vec{N}$ and $\vec{X}$ are the position vectors of the points M, N and X, respectively. On the other hand, since

$$\vec{N} = \vec{M} + \overrightarrow{MX}$$

and

$$\vec{X} = \vec{M} + \overrightarrow{MX},$$

(3) becomes

$$
\begin{aligned}
F_N &= F_M + < \overrightarrow{MN} \,,\ \vec{V} > \\
F_X &= F_M + < \overrightarrow{MX} \,,\ \vec{V} > = F_N + < \overrightarrow{NX},\ \vec{V} >
\end{aligned} \tag{4}
$$

Thus one obtains

$$F_X = \frac{1}{2} \ \{F_M + F_N + < \overrightarrow{MX} + \overrightarrow{NX} \,,\ \vec{V} > \} \tag{5}$$

The case of $F_M = F_N$ is an important special case. Now, let us discuss the necessary and sufficient conditions for this case. In this special case, the ends M and N pass round equal volumes or they draw the same curve (Γ) on K′. For this case, according to equality of (4), one can write

$$< \vec{V} \,,\ \overrightarrow{MN} > = 0.$$

Hence, there is the theorem below:

THEOREM 2: During the one-parameter closed motion K/K′, $< \vec{V} \,,\ \overrightarrow{MN} > = 0$ is the necessary and sufficient condition that the two ends M, N of the moving arc go around either the same spherical curve or tow curves of equal volume.

Now, more generally, one can seek the locus of all the points of K which pass round either the spherical curve or different spherical curves of equal volume. According to the Theorem 2, for each pair of points of this sort, the directions are orthogonal to the Steiner vector $\vec{V}$ of the motion K/K'. Therefore, all of these points must lie on the same hyperplane whose normal is $\vec{V}$.

Hence, we have is the following theorem:

THEOREM 3: Consider the volumes surrounded by different points of the moving sphere K that are not all on the same great circle. For the equality of these volumes, the necessary and sufficient condition is that these points must lie on the same hyperplane whose normal is the Steiner vector $\vec{V}$ of the motion K/K'.

In the case of $F_M = F_N$, let two ends M, N of a moving arc $\widehat{MN}$ go around the same spherical closed curve (Γ) on K'. Then the spherical ring volume F between the closed curves (Γ) and (X) can be expressed as follows

$$F = F_N - F_X \quad \text{or} \quad F = F_M - F_X$$

and, according to (4)

$$F = < \vec{XN}, \vec{V} > \quad \text{or} \quad F = < \vec{XM}, \vec{V} > . \tag{6}$$

This shows that the ring volume on the sphere depends on the Steiner vector $\vec{V}$ or the closed curve (Γ) of K'.

Now, let us rewrite (6) analitically. For this we can choose any special rectangular coordinate system in K because (6) is an inner product and an inner product is independent of coordinate transformations. For example, $\widehat{MN}$ be on the great circle of $(X_1 0 X_n)$ and $M = (1,0,\ldots,0)$. Then the central angles of $\widehat{MX}$, $\widehat{XN}$ and $\widehat{MN}$ are ρ_1, ρ_2 and ρ, respectively. Thus

$$X = (\cos \rho_1, 0,\ldots,0,\sin \rho_1)$$

and

$$N = (\cos \rho, 0,\ldots,0,\sin \rho).$$

Hence (6) reduces to

$$F = 2 \left| \frac{\sin(\rho_1/2) \sin(\rho_2/2)}{\sin(\rho/2)} V_n \right| , \tag{7}$$

and, since

$$\sin\ (\rho_1/2) = \overline{MX}/2$$

$$\sin\ (\rho_2/2) = \overline{NX}/2$$

and

$$\sin\ (\rho\ /2) = \overline{MN}/2$$

(7) becomes

$$F = \left| \frac{\overline{MX}\ .\ \overline{NX}}{\overline{MN}}\ .\ V_n \right| . \tag{8}$$

Now, consider another point Y on the arc $\widehat{MN}$ such that while the point X draws its orbit (X), Y draws another orbit (Y) on the same sphere K′. The volume F′ between the curves (Γ) and (Y), according to (8), can be expressed as follows

$$F' = \left| \frac{\overline{MY}\ .\ \overline{NY}}{\overline{MN}}\ .\ V_n \right| . \tag{9}$$

Then (7) and (8) give

$$\frac{F}{F'} = \left[\frac{\overline{MX}}{\overline{MY}}\right]^2 \frac{\overline{MY}\ .\ \overline{NX}}{\overline{MX}\ .\ \overline{NY}} \tag{2.8}$$

or if λ stands for the following ratio

$$\frac{\overline{MY}\ .\ \overline{NX}}{\overline{MX}\ .\ \overline{NY}}\ ,$$

then,

$$\frac{F}{F'} = \left[\frac{\overline{MX}}{\overline{MY}}\right]^2 .\ \lambda$$

Hence one can give the following theorems.

THEOREM 4: Let the two ends M, N of a moving arc $\widehat{MN}$ with constant lenght go around the same convex simple curve (Γ) on K′. If one chooses a fixed poing X on the arc $\widehat{MN}$, X describes a closed curve (X) on the same sphere, while M and N move on the curve (Γ). The spherical ring volume F between the two closed (Γ) and (X) can be expressed by (8).

THEOREM 5: Consider a one-parameter closed spherical motion K/K′ and a fixed great circle on the moving sphere K. Choosing four arbitrary fixed points M, N, X, Y on this great circle, let two of them move on the same curve (Γ), while the other two describe different curves (X) and (Y). If the ring volume between (Γ) and (X) is F and the volume between (Γ) and (Y) is F′, then the ratio F/F′ depends only on the relative positions of these four points M, N, X, Y.

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