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ABSOLUTELY p-th POWER CONSERVATIVE MATRICES

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ABSTRACT

In this paper, we define absolutely p-th power conservative matrices and determine the sufficient conditions for a normal matrix to be absolutely p-th power conservative.

INTRODUCTION

Let $A = (a_{nk})$ be an infinite matrix and $x = (x_k)$ be an infinite sequence and X, Y be two subsets of the space of complex sequences. If the sequence $(A_n(x)) = \left(\sum_{k=0}^{\infty} a_{nk} x_k \right)$ exists (i.e. the series on the right hand side is convergent for each n) and if $(A_n(x)) \in Y$ whenever $x \in X$, then we say that A transforms X into Y.

In this paper we introduce a Banach space S_p closely related to l_p where

$$l_p = \{u = (u_k): \sum_k |u_k|^p < \infty, 0 < p < \infty\}.$$

and determine the sufficient conditions for a normal matrix to transform S_p into S_p. We recall that a normal matrix is a semi-lower matrix with non-zero diagonal.

DEFINITION AND THE MAIN RESULT

For $1 \leq p < \infty$ let us define

$$S_p = \left\{ u = (u_k): \sum_{k=1}^{\infty} k^{p-1} |u_k|^p < \infty \right\}.$$

In what follows we assume that $1 \leq p < \infty$. Then the following properties of S_p are routine to establish.

(i) S_p is a Banach space under the norm

$$\| u \|_p = \left(\sum_{k=1}^{\infty} k^{p-1} \mid u_k \mid^p \right)^{1/p}$$

(ii) $S_p \subset l_p$, (iii) $S_p \subset S_q$, $1 \leq p < q < \infty$

Definition. A matrix $C = (c_{nk})$ is called absolutely p-th power conservative if it transformas S_p into S_p, i.e., if $C \in (S_p, S_p)$.

Now we have the following

THEOREM: Let us consider the series-to-series transformation

$$\omega_n = \sum_{v=o}^{n} c_{nv} u_v$$

Suppose that the following conditions hold

$$\sum_{n=v}^{\infty} |c_{nv}| = O(1) \tag{1}$$

$$\sum_{v=1}^{n} \frac{|c_{nv}|}{v} = O(1/n) \tag{2}$$

$$|c_{no}| = O(1/n) \tag{3}$$

Then the transformation is absolutely p-th power conservative.

PROOF: Before proving the theorem we note that (2) implies that, for fixed $v \geq 1$,

$$|c_{nv}| = O(1/n)$$

For the proof of the theorem, we write

$$\omega_n = c_{no} u_o + \sum_{v=1}^{n} c_{nv} u_v = \omega^1{}_n + \omega^2{}_n, \text{ say.}$$

By Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty} n^{p-1} |\omega^i{}_n|^p < \infty, (i = 1,2).$$

Let i=1. By (3) we have

$$n^{p-1} |c_{no}|^p = O(|c_{no}|)$$

and the result follows from the case $v=o$ of (1).

Now let $i = 2$. By Hölder's inequality when $p > 1$ (and trivially when $p = 1$) we have, for $n \geq 1$

$$|\omega^2_n|^p < \left\{ \sum_{v=1}^{n} \frac{|c_{nv}|}{v} \right\}^{p-1} \left\{ \sum_{v=1}^{n} |c_{nv}| \, v^{p-1} \, |u_v|^p \right\}$$

$$\leq \frac{M}{n^{p-1}} \sum_{v=1}^{n} |c_{nv}| \, v^{p-1} \, |u_v|^p$$

by (2), where M is a constant. Hence by (1)

$$\sum_{n=1}^{\infty} n^{p-1} \, |\omega^1|^p < M \sum_{n=1}^{\infty} \sum_{v=1}^{n} |c_{nv}| \, v^{p-1} \, |u_v|^p$$

$$= M \sum_{v=1}^{\infty} v^{p-1} \, |u_v|^p \sum_{n=v}^{\infty} |c_{nv}|$$

$$= O\left(\sum_{v=1}^{\infty} v^{p-1} \, |u_v|^p \right) < \infty$$

by assumption. Thus the theorem is proved.

REMARK. The conditions of the theorem are satisfied in particular when $C = (c_{nv})$ is a conservative Hausdorff transformation (expressed in series-to-series form). Corresponding to a sequence $\mu = (\mu_v)$ the Hausdorff transformation (H, μ_v) is given by

$$t_n = \sum_{v=o}^{n} \binom{n}{v} (\triangle^{n-v} \mu_v) \, s_v$$

where $\binom{n}{v}$ denotes the ordinary binomial coefficient, and $\triangle$ is the forward difference operator defined by $\triangle \mu_v = \mu_v - \mu_{v+1}$, $\triangle^n \mu_v = \triangle (\triangle^{n-1} \mu_v)$. For other properties of Hausdorff matrices the reader may consult (Hardy 1949).

If we suppose that $t_n = \omega_o + \omega_1 + \ldots + w_n$, and $s_v = u_o + u_1 + \ldots + u_v$, then (ω_n) is given in terms of (u_v) by

$$\omega_n = \sum_{v=o}^{n} c_{nv} \, u_v$$

where

$$c_{oo} = \mu_o$$
$$c_{no} = o \ (n \geq 1)$$
$$c_{nv} = \frac{v}{n} \binom{n}{v} \triangle^{n-v} \mu_v = \binom{n-1}{v-1} \triangle^{n-v} \mu_v, \ 1 \leq v \leq n$$

Thus (3) is trivially satisfied. It is known that (H, μ_v) is conservative if and only if μ_v can be expressed in the form

$$\mu_v = \int_0^1 t^v \, dq \ (t) \tag{4}$$

where $q(t) \in BV \ (0,1)$; so we suppose that (4) holds.

Then

$$\triangle^{n-v} \mu_v = \int_0^1 t_v \ (1-t)^{n-v} \ dq(t)$$

Thus

$$\sum_{n=v}^{\infty} |c_{nv}| \leq \sum_{n=v}^{\infty} \binom{n-1}{v-1} \int_0^1 t^v \ (1-t)^{n-v} \ |dq(t)|$$

$$= \int_0^1 \left\{ \sum_{n=v}^{\infty} \binom{n-1}{v-1} t^v \ (1-t)^{n-v} \right\} |dq(t)|$$

$$= \int_0^1 |dq(t)| = O \ (1).$$

Moreover we get

$$\sum_{v=1}^{n} \frac{|c_{nv}|}{v} \leq \frac{1}{n} \sum_{v=1}^{n} \binom{n}{v} \int_0^1 t^v \ (1-t)^{n-v} \ |dq(t)|$$

$$= \frac{1}{n} \int_0^1 \left\{ \sum_{v=1}^{n} \binom{n}{v} t^v \ (1-t)^{n-v} \right\} |dq(t)|$$

$$= \frac{1}{n} \int_0^1 (1 - (1-t)^n) \ |dq(t)| \leq \frac{1}{n} \int_0^1 |dq(t)| = O \left(\frac{1}{n}\right).$$

Hence the conditions of the theorem are satisfied.

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