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A GENERALIZATION OF THE BERTRAND CURVES AS GENERAL INCLINED CURVES IN E^n

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ABSTRACT

Bertrand curves and the Inclined curves are studied in Uyan (1978) and Özdamar and Hacısalihoğlu (1975). In this paper, Bertrand curve notion considered together with the notion of the Inclined curves. Thus, we need the generalization of the inclined curves and the Bertrand curves [Guggenheimer (1963)] and we give them.

As a result, we find that the Bertrand Curves are the Inclined curve couples. On the other hand, we give the notion of Bertrand Representation and find that the Bertrand Representation is spherical.

INTRODUCTION

In this paper, $\alpha, \beta : I \longrightarrow E^n$ are C^∞ curves, and $\{V_1, \ldots, V_n\}$ and $\{V^*_1, \ldots, V^*_n\}$ are Frénet n-Frames, respectively, on these curves. In addition; k_i and k^*_i, $1 \leq i \leq n-1$, are curvature functions of α and β, respectively. Furthermore, it is known from Gluck (1966) that the covariant derivation of V_i is

$$D_{V_1} V_i = V_i' = -k_{i-1} V_{i-1} + k_i V_{i+1}, \ 1 \leq i \leq n$$

where $k_o = k_n = 0$.

The definition of Bertrand curves is following:

"If $\beta = \alpha + \lambda V_2$, $\lambda : I \longrightarrow IR$ then (α, β) are called Bertrand curves" [Hacısalihoğlu (1983)]. If α and β are Bertrand curves, then $\lambda =$ constant and $< V^*_1, V_1 > = \text{Cos}\,\theta =$ constant [Hacısalihoğlu (1983)]. Thus, β is an inclined curve whose axis is the curve α.

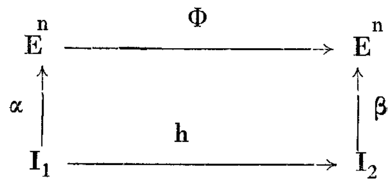
In this paper, we find a generalization of the expressions $\lambda = \text{const.}$ and $< V^*_1, V_1 > = \text{Cos}\,\theta = \text{const.}$, namely, "the distance between the corresponding points is constant" and "the angle between the corresponding tangent vectors is constant".

BERTRAND AND INCLINED CURVE COUPLES

Let α and β be two regular curves. If there exits a diffeomorphism h and a bijection Φ of E^n such that

$$B \circ h = \Phi \circ \alpha$$

then Φ is called an bijection berween the curves α and β (Fig. 1).



(Fig. 1)

The point β (h(s)) is called the corresponding point to the point α (s) under the bijection Φ.

DEFINITION 1: Let α and β be two regular curves and Φ is a bijection between them such that $\beta \circ h = \Phi \circ \alpha$. If the angle between the tangent vectors to α and β at the corresponding points is constant, then the couple (α, β) is called Inclined Curve Couple and α is called the Inclination Axis.

LEMMA 1: If the couple (α, β) is an inclined curve couple and α is the inclination axis, then the (β, α) is also inclined curve couple and the inclination axis is the curve β.

PROOF: If the inclination axes of (α, β) is the curve α then by the definition, we have

$$\beta \circ h = \Phi \circ \alpha \tag{1}$$

where h is a diffeomorphism and Φ is a bijection. So there exits a diffeomorphism h^{-1} and a bijection Φ^{-1} such that

$$h \circ h^{-1} = \text{identity}, \quad \Phi \circ \Phi^{-1} = \text{identity}.$$

Thus, from the equation (1) we may write

$$\alpha \circ h^{-1} = \Phi^{-1} \circ \beta$$

which shows that the curve couple (β, α) is an inclined curve couple and whose axis is the curve β. Q.E.D.

In the Forthcoming part of this paper, we use the expression "(α, β) is an inclined curve couple" in place of the expression "the curve couple (α, β) is an inclined curve couple, whose axis is α", since it does not make any difference whether (α, β) is an inclined curve couple or (β, α) is. Thus we don't care which curve is the inclination axis.

DEFINITION 2: The curve couple (α, β), such that $\beta = \alpha + \lambda$, is called an (r,m)-Bertrand curve couple if

$$\lambda = \sum_{i=r}^{r+m} \lambda_i V_i = \sum_{i=r}^{r+m} \lambda^*_i V^*_i \ , \ m < n-1,$$

such that, $\lambda_i, \lambda^*_i \in C^\infty (I, \mathbb{R})$, $r \leq i \leq r+m$, where V_i and V^*_i $1 \leq i \leq n$, are the Frenet vectors of the curve α and the curve β, respectively and $C^\infty (I, \mathbb{R})$ is the $\mathbb{R}$-module of the C^∞ differentiable functions from the interval I, the domain of the curves α and β, to $\mathbb{R}$. The function $\tilde{\lambda} : I \longrightarrow E^m$, $\tilde{\lambda} = (\lambda_r, \ \ldots \ , \lambda_{r+m})$ is called (r,m)-Bertrand representation of the (r,m)-Bertrand curve couple (α, β).

About Bertrand representation and Bertrand curve couple, we give the following theorem:

THEOREM 1: If (α, β) is an (r,m)-Bertrand curve couple, then the distance between the points $\alpha(s)$ and $\beta(s)$ is constant. Furthermore, the (r,m)-Bertrand representation $\tilde{\lambda}$ is a spherical curve in E^m.

PROOF: Since (α, β) is an (r,m)-Bertrand curve couple, we write $\beta = \alpha + \lambda$ and by differentiation,

$$\frac{ds^*}{ds} V^*_1 = V_1 + \sum_{i=r}^{r+m} \lambda'_i V_i - \sum_{i=r}^{r+m} \lambda_i k_{i-1} V_{i-1} + \sum_{i=r}^{r+m} \lambda_i k_i V_{i+1} \tag{2}$$

where s^* and s are the arc-length parameter of the curves β and α, respectively, and k_i, $1 \leq i \leq n-1$, is the curvature function of α and (′) denotes the derivation with respect to the parameter s, and V_i, $1 \leq i \leq n$, are the Frenet vectors of the curve α.

From Definition 2, we deduce that $V^*_1 \notin Sp\{V_r, \ldots, V_{r+m}\}$. So, by considering the equation (2), we get the following

$$\left.\begin{aligned} \lambda'_r &= \lambda_{r+1}\, k_r \\ &\vdots \\ \lambda'_i &= -\lambda_{j+1}\, k_j + \lambda_{j-1}\, k_{j-1} \\ &\vdots \\ \lambda'_{r+m} &= \lambda_{r+m-1}\, k_{r+m-1} \end{aligned}\right\} \cdots\cdots \qquad (3)$$

which is a system of first order linear differential equations. (3) has a unique solution by the initial conditions

$$\beta(s_o) = \alpha(s_o) + \lambda(s_o)\,,\ s_o \in I$$

or

$$\lambda_i(s_o) = <\beta(s_o) - \alpha(s_o)\,,\ V_i(s_o)>\,,\ r \leq i \leq r+m$$

(For the details of the unique solution, see for example [Wilson (1971)]. This unique solution is

$$\lambda(s) = e^{sA}\, C_o$$

where

$$A = \begin{bmatrix} 0 & k_r & 0 & \cdots & 0 \\ -k_r & 0 & k_{r+1} & \cdots & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & -k_{r+m} & 0 \end{bmatrix}$$

$$C_o = \begin{bmatrix} \lambda_r(s_o) \\ \vdots \\ \lambda_{r+m}(s_o) \end{bmatrix},$$

and e^{sA} denotes the exponential matrix of the matrix (sA).

On can easily show that the matrix A is an anti-symmetric matrix. Thus, the matrix e^{sA} is an orthogonal matrix [Wilson (1971)]. So, the norm of $\lambda(s)$ is same as the norm of C_o. Because C_o is constant, then the norm $\| \lambda(s) \|$ is constant. Thus,

$$\begin{aligned} d(0,\ \lambda(s)) &= \| \lambda(s) \| \\ &= \text{constant} \end{aligned}$$

that is, $\lambda(s)$ is on the origin-centered sphere in E^m, which completes the proof about the (r,m)-Bertrand representation.

On the other hand, the distance $d(\beta(s), \alpha(s))$ between the points $\beta(s)$ and $\alpha(s)$ has the following expression

$$\begin{aligned} d(\beta(s), \alpha(s)) &= \| \beta(s) - \alpha(s) \| \\ &= \| \lambda(s) \| \\ &= \text{constant} \end{aligned}$$

and that completes rest of the proof. Q.E.D.

THEOREM 2: If (α, β) is an (r,m)-Bertrand Curve Couple, then (α, β) is an inclined curve couple.

PROOF: We will prove the theorem step by step. For this, we consider first the critical cases which are $r=1,2,n-1,n$. We will then give the rest of the proof for $r \neq 1,2,n-1,n$.

Before beginning the proof let's give the following useful eguation that can be deduced from Definition 2 as in (2):

$$\frac{ds^*}{ds} V^*_1 = V_1 + \sum_{i=r}^{r+m} \lambda'_i V_i - \sum_{i=r}^{r+m} \lambda_i k_{i-1} V_{i-1} + \sum_{i=r}^{r+m} \lambda_i k_i V_{i+1} \tag{4}$$

a) The case of $r=1$ and $m=0$:

In this case, $V_1 = V_1^*$, so the angle between the α and β is constant and is egual to 0. Thus (α, β) is an inclined curve couple, where h and Φ are the identity maps in Definition 1.

b) The case $r=1$ and $m \neq 0$.

In this case, because of $V^*_1 \in Sp\{V_1, \ldots, V_{m+1}\}$, we get from the equation (4) that

$$\lambda_{m+1} = 0$$

which means that

$$V^*_1 \in Sp\{V_1, \ldots, V_m\}$$

and carrying out in the same way, we can find out that

$$V^*_1 \in Sp\{V_1\}$$

or

$$V^*_1 = V_1$$

which completes the proof for the case b).

c) The case r=2 and m=0.:

In this case, from (4) we deduce that

$$V^*_1 = \cos\theta \, V_1 + \sin\theta \, V_3$$

and by differentiation

$$\frac{ds^*}{ds} k^*_1 V^*_2 = (-\sin\theta)\,\theta' \, V_1 + (\cos\theta \, k_1 - k_2 \sin\theta) \, V_2 + \cos\theta . \theta' . V_3 + k_3 \sin\theta \; V_4.$$

From this equation one can deduce the following

$$(-\sin\theta)\,\theta' = 0$$
$$(\cos\theta)\,\theta' = 0$$
$$(\sin\theta)\,k_3 = 0$$

or

$$\theta = \text{constant}$$

which means that (α, β) is an inclined curve couple.

d) The case r=2 and $m \neq 0$.

In this case, from (4) we can write $V^*_1 \in Sp \{V_1, V_{m+3}\}$ or

$$V^*_1 = \cos\theta \, V_1 + \sin\theta \, V_{m+3}$$

and by the same way as in the case c) we get that θ = constant, thus the angle between the V^*_1 and V_1 is constant.

e) The case r=n—1 and m=0 $(n \neq 3)$

In this case, (4) becomes

$$\frac{ds^*}{ds} V_1 = V_1 + \lambda' \, V_{n-1} + \lambda \, (-k_{n-2} \, V_{n-2} + k_{n-1} \, V_n)$$

and again by differentiation with respect to s we find that

$$\lambda \, k^2_{n-1} = 0$$

Thus we have $\lambda = 0$ or $\alpha = \beta$ which completes the proof in the case e).

f) The case r=n—1, m=2.

In this case, (4) becomes

$$\frac{ds^*}{ds} V^*_1 = V_1 + (-\lambda_{n-1} \, k_{n-2}) \, V_{n-2} + (\lambda'_{n-1} - \lambda_n \, k_{n-1}) \, V_{n-1} + (\lambda'_n + \lambda_{n-1} \, k_{n-1}) \, V_n$$

that implies

$$V^*_1 \in Sp \{V_1, V_{n-2}\}$$

This is the same as in the case d) that verifies the assertation of the theorem.

g) The case $r=n$.

In this case m must be 0, and (4) implies that $\beta = \alpha$.

h) The case: $r \neq 1,2,n-1,n$.

First of all, let's examine the case $m=0$. In this case (4) implies that

$$V^*_1 \in Sp \{V_1, V_{r-1}, V_{r+1}\}$$

or

$$V^*_1 = a_1 V_1 + a_2 V_{r-1} + a_2 V_{r+1}$$

and by differentiation with respect to s, we find that

$$a_2 k_{r-1} = 0$$
$$a_3 k_r = 0$$

which means that $a_2 = a_3 = 0$, that, is, $V_1 = V^*_1$.

Secondly, if $m \neq 0$ then as before

$$V^*_1 = a_1 V_1 + a_2 V_{r-1} + a_3 V_{r+m+1}$$

and hence we write

$$a_2 k_{r-1} = 0$$
$$a_3 k_{r+m} = 0$$

which means that $a_2 = a_3 = 0$, that is, $V_1 = V^*_1$.

Both cases imply that (α, β) is an inclined curve couple. Q.E.D.

2. A Special Case, $n=3$:

Now, suppose that $n=3$, so there are five kind of Bertrand curve couples, which are (1,0), (1,1), (2,0), (2,1) and (3,0)-Bertrand curve couples. Let us explain them as follows:

a) The (1,0)-Bertrand Curve Couples: In this case $r=1$ and $m=0$, thus we have $\beta = \alpha + \lambda V_1$. From this equation, by differentiating, we obtain that

$$\frac{ds^*}{ds} V^*_1 = (1 + \lambda') V_1 + \lambda k_1 V_2$$

or $\lambda k_1 = 0$ and from this we can write $\lambda = 0$ since $k_1 \neq 0$ and $Sp \{V_1\} = Sp \{V^*_1\}$. This implies $\alpha = \beta$; that is to say, (α, β) is an inclined curve couples.

b) The (1,1)-Bertrand Curve Couples: In this case we have $\beta = \alpha + \lambda_1 V_1 + \lambda_2 V_2$. By differentiating this equation with respect to s and considering that $Sp \{V_1, V_2\} = Sp \{V^*_1, V^*_2\}$, one can easily find that $\lambda_2 = 0$. Thus $\beta = \alpha + \lambda_1 V_1$, which is the same as the case a).

c) The (2,0)-Bertrand Curve Couples: In this case, $\beta = \alpha + \lambda V_2$. This is the case that well-known definition of Bertrand Curves in E^3, [Hacısalihoğlu (1983)].

d) The (2,1)-Bertrand Curve Couples: In this case $\beta = \alpha + \lambda_1 V_2 + \lambda_3 V_3$. Since $Sp \{V_2, V_3\} = Sp \{V^*_2, V^*_3\}$ we can write $V_1 = V^*_1$. Thus (α, β) is an inclined curve couple. On the other hand, in this oase the vector field $Y = \lambda_2 V_2 + \lambda_3 V_3$ is a parallel vector field along the curve α.

e) The (3,0)-Bertrand Curve Couples: In this case $\beta = \alpha + \lambda V_3$. By differentiating this equation with respect to s and considering that $Sp \{V_3\} = Sp \{V^*_3\}$ one can easily find that $\lambda = 0$; that is, $\alpha = \beta$.

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