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UMBRELLA MATRICES AND HIGHER CURVATURES OF A CURVE

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ABSTRACT

In this paper, using the curvature matrix of a curve-hypersurface pair in the Cayley formula we obtain an umbrella matrix. Furthermore we give a relation between the Darboux matrix of the umbrella motion and the (higher) curvature matrix. In addition, using this umbrella matrix we also obtain an infinitesimal umbrella matrix.

BASIC CONCEPTS

Let M be a hypersurface in E^n and α be a curve, with the unit tangent vector field Y_1, which lies on M. Let the system $\{Y_1, \ldots, Y_{n-1}\}$, such that

$$Y_i = \bar{D}_{Y_1} Y_{i-1}\ , \ 1 < i \leq n-1$$

be linearly independent, where $\bar{D}$ is natural connection on M. Then the orthonormal system $\{X_1, \ldots, X_{n-1}\}$, which is obtained by Gram-Schmidt process from $\{Y_1, \ldots, Y_{n-1}\}$, is called the Frenet frame field of the curve α in M. If we denote the unit normal vector field to M by X_n, then the orthonormal system $\{X_1, \ldots, X_{n-1}, X_n\}$ is called the natural frame field for the curve-hypersurface pair (α, M) [Hacısalihoğlu 1983].

DEFINITION 1: Let M be a hypersurface in E^n and α be a curve on M. Then the function

$$k_{ig} : I \longrightarrow \mathbb{R}\ ,$$

given by

$$k_{ig}(s) = \langle X'_i(s)\ , X_{i+1}(s) \rangle\ ,$$

is called the i^{th}, $1 \leq i < n-1$, geodesic curvature function of the curve α and $k_{ig}(s)$ is called i^{th} geodesic curvature of α at $\alpha(s)$ for $s \in I$, where $I \subset R$ [Guggenheimer (1963)].

THEOREM 1: Let M be a hypersurface in E^n and α be a curve on M. The derivative formulas of the natural frame field $\{X_1, \ldots, X_{n-1}, X_n\}$ are

$$D_{X_1}X_i = X'_i = -k_{(i-1)g}X_{i-1} + k_{ig}X_{i+1} + II(X_1, X_i)X_n,$$

$$D_{X_1}X_n = -II(X_1,X_1)X_1 - II(X_1,X_2)X_2 - \ldots - II(X_1,X_{n-1})X_n$$

where $1 \leq i \leq n-1$ and $k_{og} = k_{(n-1)g} = 0$ [Guggenheimer (1963)].

In the matrix form, these derivative formulas become

$$\begin{bmatrix} X'_1 \\ X'_2 \\ \cdot \\ \cdot \\ \cdot \\ X'_{n-1} \\ X'_n \end{bmatrix} = \begin{bmatrix} 0 & k_{1g} & 0 \ldots & 0 & 0 & II(X_1,X_1) \\ -k_{1g} & 0 & k_{2g} \ldots & 0 & 0 & II(X_1,X_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \ldots & -k_{(n-2)g} & 0 & II(X_1,X_{n-1}) \\ -II(X_1,X_1) & \ldots & \ldots & \ldots & -II(X_1,X_{n-1}) & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_{n-1} \\ X_n \end{bmatrix}$$

or simply

$$X' = K(X)\, X \,.$$

The matrix K(X) is called the (higher) curvature matrix of the pair (α, M) [Guggenheimer (1963)] .

Let y and x be the position vectors, represented by column matrices, of a point P in the fixed space Σ^n and the moving space E^n, respectively. A continuous series of displacements, given by

$$y = Ax + b \,,$$

where the orthogonal matrix A and the translation vector b are functions of a parameter *s* which may be identified with the time, is called a motion.

Now we consider the rotational motion, given by

$$y = Ax \ .$$

The matrix

$$W = A' A^t$$

is called the angular velocity matrix or the Darboux matrix of the motion [Bottema & Roth (1979)].

DEFINITION 2: The orthogonal matrix A such that

$$AS = S$$

is called an umbrella matrix, where $S = [1\ 1\ .\ .\ .\ 1]^t \in IR_1^n$ [Özdamar (1977)].

DEFINITION 3: Let A be an umbrella matrix. The motion generated by the transformation

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

or

$$y = Ax + C$$

is called an umbrella motion in E^n, where $x,y \in IR_1^n$ and $C \in IR_1^n$ is the displacement vector of the origin [Hacısalihoğlu (1977)] .

Let $B = [b_{ij}]$ be any nxn matrix and ε be an infinitesimal quantity of the first order. Then the matrix

$$A = I_n + \varepsilon\, B$$

is called an infinitesimal matrix, where by I_n, we denote the nxn unit matrix.

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In this section we assume that the directions $X_2, X_3, \ldots, X_{n-2}$ of the natural frame field $X = \{X_1, \ldots, X_n\}$ are conjugate directions with the tangent direction X_1 for a curve α which is different from the line of curvature on a hypersurface M in E^n. Then the higher curvature matrix can be written in the form

$$K(X) = \begin{bmatrix} 0 & k_{1g} & 0 & \ldots & 0 & II(X_1,X_1) \\ -k_{1g} & 0 & k_{2g} & \ldots & 0 & 0 \\ 0 & -k_{2g} & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & II(X_1,X_{n-1}) \\ -II(X_1,X_1) & 0 & 0 & \ldots & -II(X_1,X_{n-1}) & 0 \end{bmatrix} \quad (1)$$

since

$$II(X_1,X_2) = II(X_1,X_3) = \ldots = II(X_1,X_{n-2}) = 0 \, .$$

Let us write

$$k_{ig} = b_{i+2} \qquad (1 \leq i \leq n-2)$$

$$II(X_1,X_{n-1}) = b_1$$

$$-II(X_1,X_1) = b_2 \, .$$

Thus the equality (1), with respect to the elements b_j $(1 \leq j \leq n)$, takes the form

$$K(X) = \begin{bmatrix} 0 & b_3 & 0 & 0 & \ldots & 0 & 0 & -b_2 \\ -b_3 & 0 & b_4 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -b_4 & 0 & b_5 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 0 & b_n & 0 \\ 0 & 0 & 0 & 0 & \ldots & -b_n & 0 & b_1 \\ b_2 & 0 & 0 & 0 & \ldots & 0 & -b_1 & 0 \end{bmatrix} \quad (2)$$

Let A be any nxn orthogonal matrix one of the characteristic values of which is not —1. Then A can always be expressed as

$$A = (I_n - B)^{-1} (I_n + B) , \tag{3}$$

where B is as nxn skew matrix. The formula (3) is known as the Cayley formula. Now taking K(X) instead of B in this formula we obtain the following theorem.

THEOREM 1: For $n \geq 3$, if $b_1 = b_2 = \ldots = b_n = c$ then the orthogonal matrix

$$A = (I_n - K(X))^{-1} (I_n + K(X)) \tag{4}$$

one of the charaeteristic values of which is not—1, is an umbrella matrix.

PROOF: In order to prove the theorem it suffices to show the equality

$$A\ S = S\ ,$$

where $S = [1\ 1\ .\ .\ .\ 1]^t \in \mathbb{R}^n_1$.

$$(I_n + K(X))\, S = S + K(X)\, S$$

$$= \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & -1 \\ -1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & -1 & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 - 1 \\ -1 + 1 \\ \cdot \\ \cdot \\ \cdot \\ -1 + 1 \\ 1 - 1 \end{bmatrix}$$

$$= S\ .$$

This means that S is in the kernel of K(X). Thus we have

$$(I_n - K(X))\, S = S$$

from which we get

$$S = (I_n - K(X))^{-1} S \,.$$

Then

$$\begin{aligned} AS &= (I_n - K(X))^{-1} (I_n + K(X))\, S \\ &= (I_n - K(X))^{-1} S \\ &= S \,, \end{aligned}$$

which completes the proof.

Now we may give a relation between the Darboux matrix of the umbrella motion and the higher curvature matrix by the following theorem.

THEOREM 2: Let W(A) be the Darboux matrix of the umbrella motion, where A is given by (4), and K(X) be the higher curvature matrix. Then

$$W(A) = \frac{2c'}{c} (I_n - K(X))^{-1} K(X) (I_n + K(X))^{-1} \,, \tag{5}$$

where $c = c(s)$.

PROOF: Differentiating (II.4), with respect to s, we have

$$\begin{aligned} A' \; & [(I_n - K(X))^{-1}]' (I_n + K(X)) + (I_n - K(X))^{-1} (I_n + K(X))' \\ &= -(I_n - K(X))^{-1} (I_n - K(X))' (I_n - K(X))^{-1} (I_n + K(X)) + \\ &\qquad (I_n - K(X))^{-1} K'(X) \\ &= (I_n - K(X))^{-1} K'(X) (I_n - K(X))^{-1} (I_n + K(X)) + \\ &\qquad (I_n - K(X))^{-1} K'(X) \\ &= (I_n - K(X))^{-1} K'(X) \,[\, (I_n - K(X))^{-1} (I_n + K(X)) + I_n \,] \,. \end{aligned}$$

In the last expression if we write

$$I_n = (I_n - K(X))^{-1} (I_n - K(X))$$

then we also have

$$A' = 2\, (I_n - K(X))^{-1} K'(X) (I_n - K(X))^{-1} \,.$$

From

$$K(X) = cU \qquad \text{(U is constant matrix)} \,,$$

$$K'(X) = \frac{c'}{c} K(X)$$

it follows that

$$A' = \frac{2c'}{c} (I_n - K(X))^{-1} K(X) (I_n - K(X))^{-1} .$$

Thus from $W(A) = A' A^t$ we obtain

$$W(A) = \frac{2c'}{c} (I_n - K(X))^{-1} K(X) (I_n - K(X))^{-1} (I_n + K(X))^{-1} (I_n - K(X))$$

$$= \frac{2c'}{c} (I_n - K(X))^{-1} K(X) [(I_n + K(X)) (I_n - K(X))]^{-1} (I_n - K(X))$$

$$= \frac{2c'}{c} (I_n - K(X))^{-1} K(X) [(I_n - K(X)) (I_n + K(X))]^{-1} (I_n - K(X))$$

$$= \frac{2c'}{c} (I_n - K(X))^{-1} K(X) (I_n + K(X))^{-1} (I_n - K(X))^{-1} (I_n - K(X))$$

$$= \frac{2c'}{e} (I_n - K(X))^{-1} K(X) (I_n + K(X))^{-1} .$$

THEOREM 3: Let W(A) be the Darboux matrix of the umbrella motion, where A is given by (4). Then the matrix $I_n + W(A)\,ds$ is also an (infinitesimal) umbrella matrix.

PROOF: We can write

$$Y + Y'\,ds = AX + C + (A'X + C')\,ds$$

for an infinitesimal motion in E^n. Thus since

$$X = A^t (Y - C)$$

we have

$$Y + Y'ds = (I_n + A'A^t\,ds)(Y - C) + C + C'\,ds$$

or by $W(A) = A'A^t$

$$(Y' - C')\,ds = (I_n + W(A)ds)(Y - C) - (Y - C) .$$

Moreover since

$$(I_n + W(A)ds) - I_n = W(A)\,ds$$

the matrix $I_n + W(A)$ ds is an infinitesimal orthogonal matrix. Thus for $S = [\,1\ 1\ \ldots\ 1\,]^t \in \mathbb{R}^n_1$ we get

$$
\begin{aligned}
(I_n + W(A)ds)\,S &= S + W(A)ds\,S \\
&= S + \frac{2c'}{c}(I_n - K(X))^{-1}K(X)\,(I_n + K(X))^{-1}\,S \\
&= S + \frac{2c'}{c}(I_n - K(X))^{-1}K(X)\,S.
\end{aligned}
$$

In the last expression we can write

$$K(X)\,S = 0$$

since S is in the kernel of K(X), and so we obtain

$$(I_n + W(A)ds)\,S = S$$

which means that the matrix $I_n + W(A)$ds is an (indifinitesimal) umbrella matrix.

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