Commun. Fac. Sci. Univ. Ank., Series A_1
V. 35, pp. 15 - 18 (1986)

ON THE STRONGLY REGULAR DUAL SUMMABILITY METHODS OF THE SECOND KIND

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(Received, May 20, 1986)

ABSTRACT

In this paper, the strong regularity of the series-to-series matrix transformation has been defined. Next, two theorems which are related to the strong regularity of dual summability methods of the second kind and the necessary and sufficient conditions for the strong regularity of a series-to-series matrix transformation are stated and proved. Finally, a numerical example about the dual summability methods of the second kind is given.

INTRODUCTION

Let B be a series-to-series matrix transformation which transforms the series Σu_k to the series Σx_n where,

$$x_n = \sum_{k=o}^{\infty} b_{nk} u_k , n=0,1,2,\ldots. \tag{1}$$

The necessary and sufficient conditions for the regularity of the method B are well known [Vermes (1947)].

Let us define the matrix $A = (a_{nk})$ by

$$a_{nk} = \sum_{i=o}^{n} b_{ik} \quad (\text{or equivalently } b_{nk}=a_{nk}-a_{n-1,k}). \tag{2}$$

Let A also be a summability method given by the series-to-sequence matrix transformation

$$y_n = \sum_{k=o}^{\infty} a_{nk} u_k , n=0,1,2,\ldots \quad . \tag{3}$$

The existence of (1) implies the existence of (3). The converse is also valid.

In this paper, we call the methods A and B as "dual summability methods of the second kind" and assume that these matrices are defined over complex field. It is known that A is regular if and only if B is regular [Vermes (1947)]

STRONG REGULARITY

It is said that the series $\Sigma\, u_k$ almost converges to s if its sequence of partial sums almost converges to s. This is denoted by F-lim $\Sigma\, u_k = s$, and the set of almost convergent series is also denoted by γ_F.

The strong regularity of a series-to-sequence transformation is defined in ÖZTÜRK [1983]. Let us define the strong regularity of the series-to-series matrix transformation.

DEFINITION 1. Assume that the series $\Sigma\, u_k$ almost converges to s. If the method B sums the series $\Sigma\, u_k$ to the same value s then B is called strongly regular series-to-series matrix transformation.

Let us now give the following lemma due to ÖZTÜRK [1983].

LEMMA: A regular matrix $A = (a_{nk})$ is strongly regular if and only if it satisfies the following conditions:

i) $\lim\limits_{k} a_{nk} = 0$, for each n

ii) $\lim\limits_{n} \sum\limits_{k=1}^{\infty} |\triangle^2 a_{nk}| = 0$, where $\triangle^2 a_{nk} = \triangle(a_{nk} - a_{n,k+1})$.

Now we shall give a theorem about the strong regularity of dual summability methods of the second kind.

THEOREM 1: Let A and B be the dual summability methods of the second kind. Then A is strongly regular if and only if B is strongly regular.

PROOF: **Necessity.** Let us suppose that A is strongly regular. In this case, with the notation of (3)

$$\lim y_n = \text{F-lim}\, \Sigma\, u_k$$

is valid for every almost convergent series $\Sigma\, u_k$. Then,

$$\text{F-lim} \sum u_k = \lim_n \sum_{k=o}^{\infty} a_{nk} u_k = \lim_n \sum_{i=o}^{n} \sum_{k=o}^{\infty} b_{ik} u_k \, .$$

Since the limit on the left side exists, the last limit should also exist. Hence the method B is strongly regular.

On the other hand the sufficiency is clear.

THEOREM 2: A regular matrix $B = (b_{nk})$ is strongly regular if and only if the following conditions are satisfied:

i) $\lim_k \sum_{i=o}^{n} b_{ik} = 0$, $n=0,1,2,\dots$

ii) $\lim_n \sum_{k=o}^{\infty} \left| \sum_{i=o}^{n} \triangle^2 b_{ik} \right| = 0$.

PROOF: Necessity. Assume that, B be a strongly regular series-to-series method. Then, with the notation of (1)

$$\text{F-lim} \sum u_k = \sum x_k \text{ , for each } u \in \gamma_F \, .$$

Let us suppose that (i) is not satisfied for some n. That is,

$$\lim_k \sum_{i=o}^{n} b_{ik} \neq 0 \text{ , for some } n \, .$$

Then, we can find some $u \in \gamma_F$ such that $\sum x_k$ is not convergent. Indeed, if we choose $\sum u_k = \sum (-1)^k$ then $\text{F-lim} \sum (-1)^k = 1/2$. Nevertheless the series

$$\sum x_i = \lim_n \sum_{i=o}^{n} \sum_{n=o}^{\infty} (-1)^k b_{ik}$$

diverges since $\lim_k b_{ik} \neq 0$. However this conradicts the strong regularity of B. Hence (i) is necessary.

Additionally,

$$\lim_n \sum_{k=o}^{\infty} \left| \sum_{i=o}^{n} \triangle^2 b_{ik} \right| = \lim_n \sum_{k=o}^{\infty} \left| \triangle^2 a_{nk} \right| . \tag{4}$$

Since $u \in \gamma_F$, the matrix $A=(a_{nk})$ which is related to the matrix $B=(b_{nk})$ with (2) is strongly regular by Theorem 1. Therefore, the limit tends to zero in (4) by the lemma. This proves the necessity of (ii).

Sufficiency. Let us suppose that the conditions (i) and (ii) are satisfied. For each $u \in \gamma_F$,

$$\lim_n \sum_{i=0}^{n} \sum_{k=0}^{\infty} b_{ik} u_k = \lim_n \sum_{k=0}^{\infty} a_{nk} u_k .$$

It is clear that the matrix $A=(a_{nk})$, which is dual of the second kind of the matrix $B=(b_{nk})$, satisfies the conditions of Lemma, since $B=(b_{nk})$ satisfies the conditions (i) and (ii). Hence Theorem 1 implies that the matrix $B=(b_{nk})$ is strongly regular.

This completes the proof of the theorem.

EXAMPLE: Now we shall give a numerical example about the dual summability methods of the second kind. Furthermore we shall apply these matrices to the series $\Sigma (-1)^k$.

If we define the matrix $A=(a_{nk})$ by

$$a_{nk} = \begin{cases} 1-\dfrac{(k-1)\,k}{n\,(n+1)} & , \text{for } 1 \leq k \leq n \\ 0 & , \text{for } k > n \end{cases}$$

then we get

$$b_{nk} = \begin{cases} \dfrac{2k\,(k-1)}{(n-1)\,n\,(n+1)} & , \text{for } 2 \leq k < n \\ \dfrac{2}{n+1} & , \text{for } k = n \\ 0 & , \text{for } k > n \end{cases} ,$$

by (2).

It may be easily shown that these matrices are dual of the second kind and strongly regular. Besides this, the series $\Sigma (-1)^k$ is limited to the same value $-1/2$ by each of the methods A and B.

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