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THE PROBABILITY IN THE ENTIRE SYSTEM OF SERIES OUEUES AND THE MEASURES OF EFFECTIVENESS

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ABSTRACT

The objective of this research paper is to find $P_n(N)$ the steady-state probability that there are n units in the entire system of N-stations series queues in a general form. We obtain an implicit formula for $P_n(N)$. Then, we find an explicit formula for three different models in the heterogeneous case and deduce the homogeneous case. Also we calculate both $E_N(n)$, $E_N(m)$ the expected number of units in the system and the queue in the entire system, respectively.

INTRODUCTION

This type of work has been merely defined by Hunt [1956] in the homogeneous case of two-stations and the heterogeneous case of three-stations series queues with a single server at each station.

In this research we treat a general formula for $P_n(N)$ for the case of N-stations with any number of servers at each station. We treat an explicit formula for three different models in the heterogeneous case (i.e. different ρ_i, $i=1(1)$ N), and deduce the homogeneous case. Also we derive $E_N(n)$, $E_N(m)$ the expected number of units in both the system and the queue of the entire system of N-stations for these three models.

THE IMPLICIT FORMULA OF $P_n(N)$

The steady-state probability $P_n(N)$ of finding n units in the entire system of N-stations series queues can be found in an implicit form as

$$P_n(N) = \sum_{i=0}^{n} P_i(N-1)\ P_{n-i}(1),\ N=2\ (1), \qquad (1)$$

where $P_{n-i}(1) = P_{n-i}$ is the usual steady-state probability of any queue in the case of one station only. To prove (1) we use mathematical induction. Hunt [1956] gives a formula for the two-stations series queue case in the form:

$$P_n(2) = \sum_{i=o}^{n} P_i(1)\, P_{n-i}(1). \tag{2}$$

Also for three-stations series queue, he defines:

$$P_n(3) = \sum_{j=o}^{n} P_j(1) \sum_{i=o}^{n-j} P_i(1)\, P_{n-j-i}(1).$$

Now, to complete the prove of (1), use relation (2) to get:

$$P_n(3) = \sum_{j=o}^{n} P_j(1)\, P_{n-j}(2).$$

Let $j=n-i$, then we obtain:

$$P_n(3) = \sum_{i=o}^{n} P_i(2)\, P_{n-i}(1). \tag{3}$$

Thus the relation (1) is true for $N=3$. And so if we assume that relation (1) is true for $N=k$, it could be easily proved by mathematical induction that it is true for $N=k+1$ and therefore it is true for all values of N.

THE EXPLICIT FORMULA OF P_n (N)

In this section we give the explicit formula of P_n (N) in the heterogeneous case and then deduce the homogeneous case. Also we derive E_N (n) and E_N (m) the expected number of units in both the system and the queue respectively in the entire system. We analyze the following three different models.

Model I

Consider N-stations series queue each with a single server. Let the inter-arrival times of units be an exponential with rate λ and the service times an exponential also with rates μ_i, $i=1(1)N$ at each station. The units are served according to the discipline FIFO. Then, as in Donald and Harris [1974], we write:

$$\left.\begin{aligned} &P_n(1) = P_{om} \cdot \rho_m^n, \qquad \text{and;} \\ &P_{om} = 1 - \rho_m\,,\ \ \rho_m = \frac{\lambda}{\mu_m}\ ;\ m=1(1)N. \end{aligned}\right\} \qquad (4)$$

Using relations (1) with N=2 and (4), we get:

$$P_n(2) = \sum_{i=o}^{n} P_i(1)\, P_{n-i}(1) = \left(\frac{\rho_2^{n+1} - \rho_1^{n+1}}{\rho_2 - \rho_1} \right) \prod_{m=1}^{2} P_{om}$$

From relations (1) with N=3 and (5), we obtain:

$$P_n(3) = \sum_{i=o}^{n} P_i(2)\, P_{n-i}(1)$$

$$= \left[\frac{\rho_2 (\rho_3^{n+1} - \rho_2^{n+1})}{(\rho_3 - \rho_2)(\rho_2 - \rho_1)} + \frac{\rho_1 (\rho_3^{n+1} - \rho_1^{n+1})}{(\rho_3 - \rho_1)(\rho_1 - \rho_2)} \right] \prod_{m=1}^{3} P_{om} \qquad (6)$$

Also from relations (1) with N=4 and (6), we have:

$$P_n(4) = \sum_{i=o}^{n} P_i(3)\, P_{n-i}(1)$$

$$= \left[\frac{\rho_3^2 (\rho_4^{n+1} - \rho_3^{n+1})}{(\rho_4 - \rho_3)(\rho_3 - \rho_2)(\rho_3 - \rho_1)} + \frac{\rho_2^2 (\rho_4^{n+1} - \rho_2^{n+1})}{(\rho_4 - \rho_2)(\rho_2 - \rho_3)(\rho_2 - \rho_1)} \right.$$

$$\left. + \frac{\rho_1^2 (\rho_4^{n+1} - \rho_1^{n+1})}{(\rho_4 - \rho_1)(\rho_1 - \rho_3)(\rho_1 - \rho_2)} \right] \prod_{m=1}^{4} P_{om} \qquad (7)$$

And since the relation (1) is true, it could be easily proved in general for N-stations series queues that:

$$P_n(N) = \left[\sum_{i=1}^{N-1} \left(\frac{\rho_N^{n+1} - \rho_{N-i}^{n+1}}{\rho_N - \rho_{N-i}} \right) \frac{\rho_{N-i}^{N-2}}{\prod\limits_{N-i \neq j=1}^{N-1} (\rho_{N-i} - \rho_j)} \right] \prod_{m=1}^{N} P_{om} \qquad (8)$$

In the case j=N—i the bracket of the product should be taken equal to unity, and P_{om} is given in (4).

The homogeneous case, $\rho_i = \rho$, $i=1(1)N$ could be deduced directly from (8) by taking limits as $\rho_N \to \rho_{N-1} \to \ldots \to \rho_1 = \rho$ using De L'Hospital's rule or by the following method. From relation (8) with $N=2$, take limits as $\rho_2 \to \rho_1 = \rho$ we get:

$$P_n(2) = \frac{(n+1)}{1!}(1-\rho)^2 \rho^n, \tag{9}$$

which is the same result as in Hunt [1956].
Let $N=3$ in (8), and taking limits as $\rho_3 \to \rho_2$ and $\rho_2 \to \rho_1 = \rho$, we obtain:

$$P_n(3) = \frac{(n+1)(n+2)}{2!}(1-\rho)^3 \rho^n. \tag{10}$$

By mathematical induction, we can easily generalize the relation (48) given in Jolley [1961] in the form:

$$\sum_{i=o}^{n} \prod_{j=1}^{N-1} (i+j) = \frac{1}{N} \prod_{j=1}^{N} (n+j). \tag{11}$$

Therefore using (11), we can easily generalize the relations (9) and (10) by mathematical induction in the following form:

$$\left.\begin{aligned} P_n(N) &= \frac{(1-\rho)^N \rho^n}{(N-1)!} \prod_{j=1}^{N-1} (n+j), \ n=1(1)\ldots \\ P_o(N) &= (1-\rho)^N, \ \rho = \frac{\lambda}{\mu}, \ N=2(1)\ldots \end{aligned}\right\} \tag{12}$$

The measures of effectiveness of Model I such as the expected number of units in both the entire system and the queue are:

$$E_N(n) = \sum_{n=o}^{\infty} n \, . P_n(N) = \frac{(1-\rho)^{N-2}}{(N-1)!} \prod_{j=o}^{N-1} [j-(j-1)\rho], N \geq 1 \tag{13}$$

and

$$E_N(m) = E_N(n) - N \rho^N \tag{14}$$

$N=1$ $E(n) = \frac{\rho}{1-\rho}$ one station queue only

$N=2$ $E_1(n) = \rho$ 2–stations series queue.

But as given in Hunt [1956], the expected number of units in the system of N-stations series queue is:

$$E(n) = \frac{N\rho}{1-\rho}. \tag{15}$$

Comparing $E_N(n)$ in (13) and $E(n)$ in (15), we get:

$$E(n)-E_N(n) = \frac{\rho}{(N-1)!\,(1-\rho)}\left[N!-(1-\rho)^{N-1}\prod_{j=0}^{N-1}\left\{j-(j-1)\rho\right\}\right]$$

$$> \frac{(N-1)\rho}{(1-\rho)} \geq 0, \; N=1(1)\ldots$$

i.e.,

$$E(n) \geq E_N(n).$$

The equality holds when N=1, and so we expect less units in the entire system than in the system.

Model II

Consider an N-stations series queue each with a single server and a limited capacity k. As given in Harris [1]:

$$\left.\begin{aligned} P_n(1) &= P_{om}\,\rho_m^n, \; n=1(1)k \\ P_{om} &= \frac{1-\rho_m}{1-\rho_m^{k+1}}, \; \rho_m = \frac{\lambda}{\mu_m}, \; m=1(1)N. \end{aligned}\right\} \tag{16}$$

Therefore as in Model I before, we can easily get:

$$P_n(N) = \left[\sum_{i=1}^{N-1}\left(\frac{\rho_N^{n+1}-\rho_{N-i}^{n+1}}{\rho_N-\rho_{N-i}}\right)\frac{\rho_{N-i}^{N-2}}{\prod_{N-i\neq j=1}(\rho_{N-i}-\rho_j)}\right]\prod_{m=1}^{N} P_{om} \tag{17}$$

The homogeneous case is:

$$\left.\begin{aligned} P_n(N) &= \frac{\rho^n}{(N-1)!}\left(\frac{1-\rho}{1-\rho^{k+1}}\right)^N \cdot \prod_{j=1}^{N-1}(n+j), \; n=1(1)\,k \\ P_o(N) &= \left(\frac{1-\rho}{1-\rho^{k+1}}\right)^N, \; \rho = \frac{\lambda}{\mu}, \quad N=2(1)\ldots \end{aligned}\right\} \tag{18}$$

The measures of effectiveness of Model II are:

$$E_N(n) = \sum_{n=o}^{Nk} n\, P_n(N)$$

$$= \frac{(1-\rho)^{N-2}}{(N-1)\,!\,(1-\rho^{k+1})^N} \prod_{j=o}^{N-1} [j-(j-1)\,\rho-(Nk+j+1)\,\rho^{Nk+1} + (Nk+j)\,\rho^{Nk+2}] \quad (19)$$

and $E_N(m) = E_N(n) - N\,\rho^N$

$N=1 \qquad E(n) = \dfrac{\rho\,[1-(k+1)\,\rho^k + k\rho^{k+1}]}{(1-\rho)\,(1-\rho^{k+1})}$

$N=2 \qquad E_2(n) = (1-\rho)\,[1-2\,(k+1)\,\rho^{2k+1} + (2\,k+1)\,\rho^{2(k+1)}]\,E(n)$

Model III

Consider N-stations series queues with two servers at each station (i.e each station is: M/M/2). As in Donald and Harris [1974], we have:

$$P_n(1) = \begin{cases} P_{om}\,.\,\dfrac{\rho_m^n}{n!}\,, & n=0,1\,, \\ P_{om}\,.\,2\,\rho_m^n\,, & n=2(1)\ldots \end{cases} \quad (20)$$

where $P_{om} = \dfrac{1-\rho_m}{1+\rho_m}\,,\ \rho_m = \dfrac{\lambda}{2\,\mu_m}\,,\ m=1(1)N.$

Case I: n=0,1

Using relations (1) with N=2 and (20), we get:

$$P_n(2) = \sum_{i=o}^{n} P_i(1)\,P_{n-i}(1) = \frac{2^n}{n!}\left(\sum_{i=1}^{2}\rho_i\right)^n \prod_{m=1}^{2} P_{om} \quad (21)$$

Also from relations (1), with N=3, (20) and (21), we obtain:

$$P_n(3) = \sum_{i=o}^{n} P_i(2)\,P_{n-i}(1) = \frac{2^n}{n!}\left(\sum_{i=1}^{3}\rho_i\right)^n \prod_{m=1}^{3} P_{om} \quad (22)$$

In general, by mathematical induction, it can be shown that

$$P_n(N) = \frac{2^n}{n!} \left(\sum_{i=1}^{N} \rho_i \right)^n \prod_{i=1}^{N} P_{om}, \tag{23}$$

and for the homogeneous case:

$$P_n(N) = \frac{2^n}{n!} (N\rho)^n P_o^N , \; n=0,1 \tag{24}$$

Case II: n=2(1)...

Using relations (1), with N=2, and (20), we obtain:

$$P_n(2) = P_{o2} P_n(1) + 2 \left[\rho_2^n + 2 \left(\frac{\rho_1\rho_2^n - \rho_2\rho_1^n}{\rho_2 - \rho_1} \right) \prod_{m=1}^{2} P_{om} \right. \tag{25}$$

From (1), with N=3, (20) and (25), we have:

$$P_n(3) = P_{o3} P_n(2) + 2 \left[\rho^n_3 + 2 \left(\frac{\rho_2\rho_3^n - \rho_3\rho_2^n}{\rho_3 - \rho_2} \right) \left(\frac{\rho_1 + \rho_2}{\rho_2 - \rho_1} \right) \right.$$

$$\left. + 2 \left(\frac{\rho_1\rho_3^n - \rho_3\rho_1^n}{\rho_3 - \rho_1} \right) \left(\frac{\rho_1 + \rho_2}{\rho_1 - \rho_2} \right) \right] \cdot \prod_{m=1}^{3} P_{om} \tag{26}$$

Also from relations (1), with N=4, and (20), we deduce:

$$P_n(4) = P_{o4}P_n(3) + 2 \left[\rho_4^n + 2 \left(\frac{\rho_3\rho^n_4 - \rho_4\rho^n_3}{\rho_4 - \rho_3} \right) \left\{ \frac{\rho_3^2 + \rho_3(\rho_1 + \rho_2) + \rho_1\rho_2}{(\rho_3 - \rho_2)(\rho_3 - \rho_1)} \right\} \right.$$

$$+ 2 \left(\frac{\rho_2\rho_4^n - \rho_4\rho_2^n}{\rho_4 - \rho_2} \right) \left\{ \frac{\rho_2^2 + \rho_2(\rho_1 + \rho_2) + \rho_1\rho_3}{(\rho_2 - \rho_3)(\rho_2 - \rho_1)} \right\}$$

$$\left. + 2 \left(\frac{\rho_1\rho_4^n - \rho_4\rho_1^n}{\rho_4 - \rho_1} \right) \left\{ \frac{\rho_1^4 + \rho_1(\rho_2 + \rho_3) + \rho_2\rho_3}{(\rho_1 - \rho_3)(\rho_1 - \rho_4)} \right\} \right] \prod_{m=1}^{4} P_{om} \tag{27}$$

Thus, by mathematical induction, we can easily generalize relations (25) to (27) in the form:

$$P_n(N) = P_{0N} P_n(N-1) + 2 \left[\rho_N^n + \right.$$

$$\left. 2 \sum_{i=1}^{N-1} \left(\frac{\rho_{N-i}\rho^n{}_N - \rho_N\rho^n{}_{N-i}}{\rho_N - \rho_{N-i}} \right) \frac{\alpha\,(N,i)}{\prod_{n-i \neq j=1}^{N-1} (\rho_{N-i} - \rho_j)} \right] \cdot \prod_{m=1}^{N} P_{om} \tag{28}$$

where

$$\alpha(N,i) = \begin{cases} 1 & N=2 \\ \rho_{3-i} + \rho_i & N=3 \\ \rho_{N-i}^{N-2} + \sum_{k=1}^{N-3} \rho_{N-i}^{N-k-2} (\Sigma_k) + \prod_{N-i\neq j=1}^{N-1} \rho_j, & N=4(1)\ldots \end{cases} \tag{29}$$

and Σ_k = sum of the product of ρ's taken k at a time excluding ρ_N and ρ_{N-i} from them. The homogeneous case, $\rho_i = \rho$, $i=1(1)N$ can be deduced directly from relation (28) by taking limits as before. Using relation (28), with $N=2$, and taking limits as $\rho_2 \to \rho_1 = \rho$, we get:

$$P_n(2) = P_o^2 \rho^n \left[\frac{2^2}{1!} \binom{2}{0} (n-1)_1 + 2 \binom{2}{1} \right] \tag{30}$$

From (28) with $N=3$ and taking limits as $\rho_3 \to \rho_2$, $\rho_2 \to \rho_1 = \rho$ we obtain:

$$P_n(3) = P_o^3 \rho^n \left[\frac{2^3}{2!} \binom{3}{0} (n-1)_2 + \frac{2^2}{1!} \binom{3}{1} (n-1)_1 + 2 \binom{3}{2} \right] \tag{31}$$

In general, by mathematical induction, we can easily prove that:

$$P_n(N) = P_o^N \rho^n \sum_{i=0}^{N-1} \binom{N}{i} (n-1)_{N-i-1} \frac{2^{N-i}}{(N-i-1)!}, \quad N=2(1)\ldots \tag{32}$$

Thus, from (24) and (32), we can write:

$$P_n(N) = \begin{cases} \frac{2^n}{n!} P_o^N (N\rho)^n, & n=0,1 \\ P_o^N \rho^n \sum_{i=0}^{N-1} \binom{N}{i} (n-1)_{N-i-1} \frac{2^{N-i}}{(N-i-1)!}, & n=2(1)\ldots \end{cases} \tag{33}$$

and

$$P_o = \frac{1-\rho}{1+\rho}. \tag{34}$$

For $N=1$, we get the same result given as in Harris [1]. The measures of effectiveness of Model III are:

$$E_N(n) = \sum_{n=0}^{\infty} n P_n(N) = \frac{2N\rho}{(1-\rho)^2} \left(\frac{1+\rho}{1-\rho} \right)^{N-1} \cdot P_o^N$$

$$\left.\begin{aligned} &= \frac{2N\rho}{1-\rho^2} \\ E_N(m) &= E_N(n) - N\,\rho^N \end{aligned}\right\} \qquad (35)$$

$N=1 \qquad E(n) = \dfrac{2\rho}{1-\rho^2}$ which is donald and Harris's [1974] result.

$$N=2 \qquad E_2(n) = \frac{4\rho}{1-\rho^2}\,, \ \rho = \frac{\lambda}{2\mu}\,.$$

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