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A NOTE ON COMMUTATIVITY OF RINGS

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ABSTRACT

In this note we prove that if R is a semi-prime ring with unity satisfying $(xy)^2 = y^2 x^2$, for all $x, y \in R$ then R is commutative.

INTRODUCTION

This is well-known that a group G satisfying $(xy)^2 = x^2 y^2$, for all x, y in G must be commutative. E.C. Johsen, D.L. Outcalt and Adil Yaqub [1968] proved a ring-theoretic analogue of the above result. In the present note we attempt to prove that if R is a semi-prime ring with unity satisfying $(xy)^2 = y^2 x^2$, for all $x, y \varepsilon R$, even then R is commutative. However we give an example which shows that the results is not valid for arbitrary rings.

In preparation for the proof of this theorem, we first have the following lemmas.

LEMMA 1: If R is a semi-prime ring satisfying $(xy)^2 = y^2 x^2$ for all $x, y \varepsilon R$, then R has no nonzero nilpotent element.

PROOF: Let $a \varepsilon R$ such that $a^2 = 0$. Using the hypothesis we get $(ax)^2 = 0$, for all $x \varepsilon R$. If $aR \neq 0$, then the above shows that aR is a nonzero nilright ideal satisfying the identity $y^2 = 0$ for all y in aR. So by Lemma 2.1.1 of Herstein (1976) R has a nonzero nilpotent ideal. This is a contradiction since R is semi-prime. Thus $aR = 0$, and hence $aRa = 0$. This implies that $a = 0$ since R is semi-prime.

LEMMA 2: If R is a prime ring satisfying $(xy)^2 = y^2 x^2$, for all $x, y \varepsilon R$, then R has no zero divisors.

PROOF: By Lemma 1 above, R has no nonzero nilpotent elements. So by lemma 1.1.1 of Herstein (1976), R has no zero divisors since it is prime with no nonzero nilpotent element.

MAIN RESULT

THEOREM: Let R be a semi-prime ring with unity satisfying $(xy)^2 = y^2x^2$, for all $x,y \in R$, then R is commutative.

PROOF: Since R is semi-prime ring then it is isomorphic to the subdirectsum of prime rings R_α, each of which, as a homomorphic image of R, satisfies the hypothesis placed on R. So we may assume that R is prime. On replacing y by $(1+y)$ in $(xy)^2 = y^2 x^2$, we get

$$x^2y + xyx - 2yx^2 = 0 \tag{1}$$

Case I. If Char R = 2, then from (1) we obtain $x(xy + yx) = 0$. By Lemma 2.2, it gives that if $x \neq 0$ then $xy + yx = 0$ and $x = 0$ also yields $xy + yx = 0$. Thus in every case $xy + yx = 0$, which gives $xy = yx$, as Char R = 2.

Case II. If Char R $\neq 2$, then with $y = y + y^2$ in (1) we get

$$x^2y^2 + xy^2x - 2y^2x^2 = 0. \tag{2}$$

Multiply (1) on the left by y, to get

$$yx^2y + (yx)^2 - 2y^2x^2 = 0. \tag{3}$$

From (2) and (3), we have

$$xy^2x = yx^2y, \text{ for all } x, y \in R. \tag{4}$$

Substituting $(x + y)$ for y and, simplifying we get,

$$x^3y + yx^3 - x^2yx - xyx^2 = 0. \tag{5}$$

On replacing x by $(1 + x)$, (5) gives

$$2(x^2y + yx^2 - 2xyx) = 0 \tag{6}$$

which implies $x^2y + yx^2 - 2xyx = 0$ and with $y = y + y^2$, (6) gives

$$x^2y^2 + y^2x^2 - 2xy^2x = 0 \tag{7}$$

Also (6) gives

$$\left.\begin{aligned} x^2y^2 &= 2\,(xy)^2 - yx^2y \\ \text{and}\quad y^2x^2 &= 2\,(yx)^2 - yx^2y \end{aligned}\right\} \tag{8}$$

Now from (7) and (8), we have

$$2\,(xy - yx)^2 = 0$$

which implies that $(xy - yx)^2 = 0$. Now again by Lemma 2, $xy = yx$, and R is commutative. This completes the proof of our theorem.

The following example shows that this theorem is not valid for arbitrary rings.

Example. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \;\middle|\; a,\ b \text{ are integers} \right\}$. It is easily varified that $(xy)^2 = y^2x^2$, for all $x, y \varepsilon R$. However R is not commutative.

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