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A Note On Entire Dirichlet Series Of Irregular Growth

by

SATENDRA K. VAISH

25

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A Note On Entire Dirichlet Series Of Irregular Growth

SATENDRA K. VAISH

Department of Mathematics, Pantnagar University, Pantnagar– 263145 (U.P.), India.

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ABSTRACT

For an entire Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, $(s = \sigma + it,\ 0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots,\ \lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = 0)$, the order $\rho(0 \leq \rho \leq \infty)$ and lower $\lambda(0 \leq \lambda \leq \infty)$ are given by the limit superior and limit inferior of $(\log\log M(\sigma))/\sigma$, as $\sigma \to \infty$, respectively, where $M(\sigma) = \sup\{|f(\sigma+it)|: -\infty < t < \infty\}$. In this note we derive a formula for lower proximate type $T_\lambda\{ = \lim_{\sigma\to\infty} \inf (\log M(\sigma))/e^{\sigma\lambda(\sigma)}\}$, $0 < T_\lambda < \infty$, $0 < \lambda < \infty$, in terms of the coefficients a'_ns and λ'_ns. Here $\lambda(\sigma)$ is the lower proximate order or f(s).

1. Let the function defined by $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, where $s = \sigma + it$ and $\{\lambda_n\}$ is a sequence of real numbers such taht $0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_n \uparrow \infty$ as $n \to \infty$ and $\lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = 0$, represents an entire function. The order ρ and lower order λ of f (s) are defined [1, p. 77] as:

$$(1.1)\quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix},\quad 0 \leq \lambda \leq \rho < \infty,$$

where $M(\sigma) = \sup\{|f(\sigma+it)|: -\infty < t < \infty\}$.

Let $\mu(\sigma)$ be the maximum term in the representation of $\Sigma\, |a_n| \exp(\sigma\lambda_n)$ and call it as the maximum term of f (s).

For function of order ρ $(0 < \rho < \infty)$, the type T and lower type t are given as:

$$(1.2)\quad \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \begin{matrix} T \\ , \\ t \end{matrix} \quad 0 \leq t \leq T \leq \infty .$$

An entire function is said to be of irregular growth, if $\rho \neq \lambda$. It can easily be shown that the lower type of an entire function of irregular growth of finite order is zero, that is,

$$\lim_{\sigma \to \infty} \inf \frac{\log M(\sigma)}{e^{\rho\sigma}} = 0 .$$

In such a case, if the quantity

$$(1.3)\quad \lim_{\sigma \to \infty} \inf \frac{\log M(\sigma)}{e^{\sigma \lambda(\sigma)}} = T_{\lambda}$$

is different from zero and infinity, then T_{λ} will be called the 'lower proximate type' of f (s) with respect to the comparison function $\lambda(\sigma)$, which satisfy the following conditions:

$$(1.4)\quad \lim_{\sigma \to \infty} \lambda(\sigma) = \lambda$$

$$(1.5)\quad \lim_{\sigma \to \infty} \sigma \lambda'(\sigma) = 0,$$

where $\lambda'(\sigma)$ is either the left or right hand derivative at points where they are different.

In the present paper, we obtain a formula for T_{λ} in terms of the coefficients a'_n s and λ_n' s.

We prove the following:

2. *Theorem. Let* f (s) *be an entire function of lower order* λ $(0 < \lambda < \infty)$. *Then a necessary and sufficient condition that* T_{λ} *be the 'lower proximate type' of* f (s), *is given by*

$$(2.1)\quad \lim_{n \to \infty} \inf \{F(\lambda_n) |a_n|^{1/\lambda_n}\} = (e \lambda T_{\lambda})^{1/\lambda},$$

where F (x) *is defined to be the unique solution* (*when* $x > x_0$) *of the equation*

(2.2) $x = e^{\lambda(\log F)\log F}$

and $(\log |a_n/a_{n+1}|)/(\lambda_{n+1} - \lambda_n)$ *forms a non-decreasing function of* n *for* $n > n_0$.

Proof. From (2.2), we have

$\log x = \lambda(\log F)\log F.$

Therefore, $\dfrac{d(\log x)}{d(\log F)} = \lambda'(\log F)\log F + \lambda(\log F)$

or, $\lim\limits_{F \to \infty} \dfrac{d(\log x)}{d(\log F)} = \lambda,$

by using (1.4) and (1.5). Writing this in the form

$$\lim_{x \to \infty} \frac{d(\log F(x))}{d(\log x)} = \frac{1}{\lambda},$$

or, $\left(\dfrac{1}{\lambda} - \varepsilon\right) d(\log x) < d(\log F(x)) < \left(\dfrac{1}{\lambda} + \varepsilon\right) d(\log x)$ for any

$\varepsilon > 0$ and $x > x_0$. This on integration from x to hx gives,

$$\left(\frac{1}{\lambda} - \varepsilon\right)\log h < \log\left\{\frac{F(xh)}{F(x)}\right\} < \left(\frac{1}{\lambda} + \varepsilon\right)\log h.$$

Hence,

$$(2.3)\quad \lim_{x \to \infty} \frac{F(xh)}{F(x)} = h^{1/\lambda}.$$

For functions of finite order ρ, we have [2]:

$\log M(\sigma) \approx \log \mu(\sigma)$, as $\sigma \to \infty$.

Hence, M (σ) can be replaced by $\mu(\sigma)$ in (1.1), (1.2) and (1.3) etc. Now, from (1.3), we have for $T_1 < T_\lambda$ and for all large values of σ,

$\log \mu(\sigma) > T_1 e^{\sigma\lambda(\sigma)}.$

In particular, if we take $R_n \leq \sigma < R_{n+1}$, where

$$R_{n+1} = \frac{\log |a_n/a_{n+1}|}{\lambda_{n+1} - \lambda_n}, \text{ then}$$

$$\log |a_n| > T_1 e^{\sigma\lambda(\sigma)} - \sigma \lambda_n$$

for all large values of n.

Let $\lambda_n = T_1 \lambda e^{\sigma\lambda(\sigma)}$. Then, for $n > n_0$

$$\log |a_n| > \frac{\lambda_n}{\lambda} - \lambda_n \log F\left(\frac{\lambda_n}{T_1 \lambda}\right)$$

$$\text{or, } \log \{F(\lambda_n)\ |a_n|^{1/\lambda_n}\} > \frac{1}{\lambda} + \log \left\{\frac{F(\lambda_n)}{F\left(\frac{\lambda_n}{T_1\lambda}\right)}\right\}.$$

Using (2.3) and the fact that $T_\lambda - T_1$ is arbitrary, we get

$$(2.4)\quad \liminf_{n \to \infty} \{F(\lambda_n)\ |a_n|^{1/\lambda_n}\} \geq (e\,\lambda\, T_\lambda)^{1/\lambda}.$$

We now show that the inequality in (2.4) can not occur. For in that case, a number T_2 ($T_2 > T_\lambda$) can be found such that

$$(2.4)\quad \liminf_{n \to \infty} \{F(\lambda_n)\ |a_n|^{1/\lambda_n}\} = (e\,\lambda\, T_2)^{1/\lambda}.$$

Let us choose any number T_3 between T_2 and T_λ ($T_2 > T_3 > T_\lambda$), then for $n > n_0$, we have

$$|a_n| > \left\{\frac{(e\,\lambda\, T_3)^{1/\lambda}}{F(\lambda_n)}\right\}^{\lambda_n},$$

$$\text{or,}\quad |a_n| > \left\{\frac{e^{1/\lambda}}{F\left(\frac{\lambda_n}{T_3\lambda}\right)}\right\}^{\lambda_n},$$

which with the Ritt's inequality $M(\sigma) \geq |a_n|\, e^{\sigma\lambda_n}$ gives

$$M(\sigma) > \left\{\frac{e^{1/\lambda}}{F\left(\frac{\lambda_n}{T_3\lambda}\right)}\right\}^{\lambda_n} e^{\sigma\lambda_n}.$$

Taking $\lambda_n = T_3 \lambda e^{\sigma\lambda(\sigma)}$, it follows that

$$\liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\sigma\lambda(\sigma)}} \geq T_3,$$

or, $T_\lambda \geq T_3$,

which contradicts that $T_\lambda < T_3$. This completes the proof of Theorem.

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