Commun. Fac. Sci. Univ. Ank., Series A_1
V. 35, pp. 94 - 100 (1986)

THE CESÀRO SPACE OF NULL SEQUENCES

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(Received: August 13, 1985)

ABSTRACT

FK space inclusion, weak convergence, extreme points of the unit disc and other properties of the Cesàro space of the null sequences are discussed.

INTRODUCTION

A sequence whose k-th term is x_k is denoted by $\{x_k\}$ or simply x. We employ the following notation:

$\varnothing$: the set of all finite sequences.

c_o : the set of all null sequences.

m : the set of all bounded sequences.

h : the set of all sequences x such that x is a null sequence and

$$\sum_{k=1}^{\infty} k \mid x_k - x_{k+1} \mid \text{ converges.}$$

δ^k : the sequence $\{0,0,\ldots,1,0,\ldots\}$, 1 in the k-th place and zeros elsewhere. Here, $k = 1, 2, \ldots$

$\sigma(c_o)$: the BK-space of all sequences x such that the Cesàro transform $\{k^{-1}(x_1 + x_2 + \ldots + x_k)\}$ is a null sequence. The norm on $\sigma(c_o)$ is given by

$$\| x \| = \sup_{(k)} k^{-1} \mid x_1 + x_2 + \ldots + x_k \mid$$

σ (m) : the BK-space of all sequences x such that the Cesàro transform $\{k^{-1}(x_1 + x_2 + \ldots + x_k)\}$ is a bounded sequence, with the same norm as in $\sigma(c_o)$.

The space $\sigma(c_o)$ is called the Cesàro space of the null sequences. Given a sequence space X we write

X' for the conjugate space of X;
X^α for the α-dual space of X;
X^β for the β-dual space of X;
X^γ for the γ-dual space of X;
X^f for the f-dual space of X.

Here, it must be emphasized that X should contain $\varnothing$ for X^f to be defined.

Let X be an FK-space containing $\varnothing$. Then $F^+(X)$ will be the set of all those sequences z such that the series

$$\sum_{k=1}^{\infty} z_k f(\delta^k)$$

converges for every f in X'. For further notation and terminology, we refer the reader to [Goes and Goes(1970)] and [Wilansky (1984)].

We note that $\sigma(c_0)$ is normal and hence it is monotone. Therefore,

$$[\sigma(c_o)]^\alpha = [\sigma(c_o)]^\beta = [\sigma(c_o)]^\gamma.$$

Also, $\sigma(c_o)$ has monotone norm. Hence, by theorem 10.3.12 of Wilansky (1984), it follows that $\sigma(c_o)$ has AB. Consequently,

$$[\sigma(c_o)]^f = [\sigma(c_o)]^\gamma$$

But from [Goes and Goes (1970), p. 97] we have that

$$[\sigma(c_o)]^\beta = h.$$

Thus, we conclude that

$$[\sigma(c_o)]^f = h.$$

The object of this paper is to investigate some properties of $\sigma(c_o)$.

RESULTS

PROPOSITION 1: Let X be an FK-space containing $\varnothing$. Then X contains $\sigma(c_o)$ if and only if the sequence $\{f(\delta^k)\}$ belongs to h for every f in X'.

PROOF: First, $\sigma(c_o)$ has AK. Hence, it has AD. Therefore, by theorem 8.6.1 of Wilansky (1984)

$$\sigma\ (c_o) \subset X \Leftrightarrow X^f \subset [\sigma\ (c_o)]^f = h$$

$$\Leftrightarrow \{f\ (\delta^k)\} \in h \text{ for every } f \text{ in } X'.$$

This proves the proposition.

PROPOSITION 2: Let X be an FK-space containing $\varnothing$. Then X contains $\sigma\ (c_0)$ if and only if F^+ (X) contains σ (m).

PROOF: Suppose that X contains $\sigma\ (c_o)$. Then F^+ (X) contains F^+ $(\sigma\ (c_o))$. But, by theorem 10.4.2 of Wilansky (1984) we have

$$F^+\ (\sigma\ (c_o)) = [\sigma\ (c_o)]^{f\beta} = h^\beta = \sigma\ (m)$$

Hence, F^+ (X) contains σ (m).

Conversely, suppose that F^+ (X) contains σ (m). Then, $[\sigma\ (m)]^\beta$ contains $[F^+ (X)]^\beta$. But $[\sigma\ (m)]^\beta = h$. Therefore, h contains $[F^+ (X)]^\beta$. Also, $X^f \subset X^{f\beta\beta} = [F^+\ (X)]^\beta$.

Thus, h contains X^f. But then, since h has AD, it follows that

$$\sigma\ (c_o) \subset \sigma\ (m) = h^f \subset X^{ff} \subset X$$

This completes the proof.

PROPOSITION 3: c_o is dense in $\sigma\ (c_o)$.

PROOF: Let x be any element $\sigma\ (c_o)$. Take the n-th section of x, namely,

$$x^{[n]} = \{x_1,\ x_2, \ldots, x_n, 0, 0, \ldots\}$$

Now the sequence of sequences $\{x^{[n]}\}$ is in c_o. Because $\sigma\ (c_o)$ has AK. $x^{[n]} \to x$ in $\sigma\ (c_o)$ as $n \to \infty$. Therefore, x belongs to the closure of c_o. Hence, c_o is dense in $\sigma\ (c_o)$. This proves the result.

PROPOSITION 4: $\sigma\ (c_o)$ is the largest AD - space X such that $X^{\beta\beta} = \sigma$ (m).

PROOF: Let Y be an arbitrary AD- space such that

$$h = Y^\beta \subset Y^f$$

so that

$$\sigma\ (m) = h^\beta = Y^{\beta\beta}.$$

But $[\sigma\ (c_o)] = h$. Also Y has AD. Therefore, by theorem 8.6.1 of [Wilansky (1984)], we have

$$[\sigma(c_o)]^f \subset Y^f \text{ implies that } Y \subset \sigma(c_o).$$

This establishes the proposition.

PROPOSITION 5: Weak convergence does not imply strong convergence in $\sigma(c_o)$.

PROOF: If weak convergence were to imply strong convergence in $\sigma(c_o)$ we would have by [Wilansky (1984), p. 195]

$$[\sigma(c_o)]^{\beta\beta} = \sigma(c_o).$$

But,

$$[\sigma(c_o)]^{\beta\beta} = h^\beta = \sigma(m) \neq \sigma(c_o).$$

This contradiction shows that weak convergence does not imply strong convergence in $\sigma(c_o)$, and hence the result follows.

PROPOSITION 6: Let $\triangle = \{\delta^1, \delta^2, \ldots\}$. Let $(X, |||\cdot|||)$ be a BK-space with basis $\triangle$. Then $\triangle$ is bounded away from zero in X, that is,

$$\inf_{(k)} |||\delta^k||| > 0$$

if $X \subset \sigma(c_o)$.

PROOF: Since $X \subset \sigma(c_o)$, we have that X is a BK-space having a topology stronger than $\sigma(c_o)$. Hence there exists an n such that for x in X

$$n\, |||x||| \geq \|x\|$$

Here, $\|\cdot\|$ denotes the norm on $\sigma(c_o)$. Takin $x = \delta^k$, we obtain, for each k,

$$|||\delta||| \geq 1/kn$$

Therefore, $\triangle$ is bounded away from zero in X. This completes the proof.

PROPOSITION 7: The unit disc (closed unit sphere) D in $\sigma(c_o)$ has no extreme points.

PROOF: Let $z \in$ D.Let

$$k^{-1}\, |z_1 + z_2 + \ldots + z_k| < 1$$

for some $k = k_o$. This is possible, because the sequence $\{k^{-1}(z_1 + z_2 + \ldots + z_k)\}$ is a null sequence. Let $\varepsilon > 0$ be defined by

$$\varepsilon < 1 - k_o^{-1} \mid z_1 + z_2 + \ldots + z_k \mid$$

In case $k = k_o$, we take

$$x = z + \varepsilon \, \delta^{k_o}$$

$$y = z - \varepsilon \, \delta^{k_o}$$

But then

$$k_o^{-1} \, |x_1 + x_2 + \ldots + x_{k_o}| = k_o^{-1} (|z_1 + z_2 + \ldots + z_{k_o}| + \varepsilon)$$

$$< k_o^{-1} |z_1 + z_2 + \ldots + z_{k_o}| + \varepsilon$$

$$< 1$$

so that x is in D. Similarly it can be shown that y is in D. In case $k \neq k_o$, we take

$$x = z + \varepsilon \, (\delta^{k_o} - \delta^{k_o+1})$$

$$y + z - \varepsilon \, (\delta^{k_o} - \delta^{k_o+1})$$

so that

$$k^{-1} \mid x_1 + x_2 + \ldots + x_k \mid \leq \| z \| \leq 1 \text{ for } k \neq k_o.$$

Therefore, $\| x \| \leq 1$, so that $x \in D$. A similar argument shows that $y \in D$. In either case, $z = (x + y)/2$. So, z is not an extreme point of D. Thus, D has no extreme points in $\sigma \, (c_o)$. This establishes the result.

ACKNOWLEDGEMENT

I thank Professor Brian Kuttner for his interest in this work.

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