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On The Degree Of Approximation By Bernestein Type Operators

by

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On The Degree Of Approximation By Bernstein Type Operators

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1. Introduction And Results:

If f (x) is a function defined on [0,1], the Bernstein polynomial $B_n(f;x)$ of f(x) is

$$(1.1)\qquad B_n(f;x) = \sum_{k=o}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \qquad \text{where}$$

$$(1.2)\qquad P_{n,k}(x) = \binom{k}{n} x^k (1-x)^{n-k},$$

Schurer [5] introduced the operator

$$S_{nr}: C\left[0,1+\frac{r}{n}\right] \to C\,[0,1], \text{ defined by}$$

$$(1.3)\qquad S_{nr}(f;x) = \sum_{k=o}^{n+r} f\left(\frac{k}{n}\right) p_{nr,k}(x), \qquad \text{where}$$

$$(1.4)\qquad p_{nr,k}(x) = \binom{n+r}{k} x^k (1-x)^{n+r-k}$$

and r is a non-negative integer. In case $r=0$, this reduces to the well known Bernstein operator (1.1).

A small modification of Bernstein polynomial $B_n(f;x)$ of f(x) due to Kantorovič [4] makes it possible to approximate Lebesgue integrable function in the L_1-norm by the modified polynomial

$$(1.5) \quad P_n(f;x) = (n+1) \sum_{k=o}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right) p_{n,k}(x)$$

where $p_{n,k}(x)$ is same as defined by (1.2).

Jenson [3] established the interesting generalization of bionomial theorem as:

$$(1.6) \quad (x+y+n\alpha)^n = \sum_{k=o}^{n} \binom{n}{k} x\,(x+k\alpha)^{k-1} (y+(n-k)\alpha)^{n-k},$$

which on substituting (n+r) for n, becomes

$$(1.7) \quad (x+y+(n+r)\alpha)^{n+r} = \sum_{k=o}^{n} \binom{n+r}{k} (x+k\alpha)^{k-1} (y+(n+r-k)\alpha)^{n+r-k}.$$

Substituting $y = (1-x)$, we obtain

$$(1.8) \quad 1 = \sum_{k=o}^{n+r} \frac{x(x+k\alpha)^{k-1} (1-x+(n+r-k)\alpha)^{n+r-k}}{(1+(n+r)\alpha)^{n+r}} \binom{n+r}{k}, \text{ and let}$$

$$(1.9) \quad p_{nr,k}(x;\alpha) = \binom{n+r}{k} \frac{x(x+k\alpha)^{k-1} (1-x+(n+r-k)\alpha)^{n+r-k}}{(1+(n+r)\alpha)^{n+r}}$$

$$(1.10) \quad \sum_{k=o}^{n+r} p_{nr,k}(x;\alpha) = 1$$

For the finite interval $\left[0,1 + \frac{r}{n}\right]$, we modify the operator in a manner similar to that done to Berns'tein's operator by Kantorovic [4] and define the operator

$$A_{nr}: C\left[0,1+\frac{r}{n}\right] \rightarrow C\,[0,1], \text{ by}$$

$$(1.11) \quad A^{\alpha}_{nr}(f;x) = (n+r+1) \sum_{k+o}^{n+r} \left(\int_{k/(n+r+1)}^{(k+1)/(k+r+1)} f(t)dt \right) p_{nr,k}(x;\alpha)$$

where $p_{nr,k}(x;\alpha)$ is the same as (1.9) and r is a non-negative integer. When $r=0$ and $\alpha=0$, it reduces to the well-known Kantorovic operator (1.5).

The functions

$$(1.12)\quad S(v,n+r,x,y) = \sum_{k}^{n+r} \binom{n+r}{k} (x+k\alpha)^{k+v-1} [y+(n+r-k)\alpha]^{n+r-k}$$

satisfy the reduction formula

$$(1.13)\quad S(v,n+r,x,y) = x\, S(v-1,\ n+r,x,y) + (n+r)\alpha\, S(v,n+r-1,x+\alpha,y)$$

and hence

$$(1.14)\quad x\, S(o,n+r,x,y) = (x+y+(n+r)\alpha)^{n+r}.$$

By repeated use of reduction formula (1.13) and using (1.14) we get

$$(1.15)\quad S(1,n+r,x,y) = \sum_{k=o}^{n+r} \binom{n+r}{k} k!\, \alpha^{n+r}(x+y+(n+r)\alpha)^{n+r-k}$$

and

$$(1.16)\quad S(2,n+r,x,y) = \sum_{k}^{n+r} (x+k\alpha) \binom{n+r}{k} k!\alpha^{k}\, S(1,n+r-k,x+k\alpha,y)$$

Replacing k! by $\int_{o}^{\infty} t^{k} e^{-t}\, dt$ and using binomial theorem, we obtain

$$(1.17)\quad S(1,n+r,x,y) = \int_{o}^{\infty} e^{-t} (x+y+(n+r)\alpha + t\alpha)^{n+r}\, dt$$

and

$$(1.18)\quad S(2,n+r,x,y) = \int_{o}^{\infty} e^{-t}\, dt \int_{o}^{\infty} e^{-s}\, ds\, \{ (x+y+(n+r)\alpha+t\alpha+s\alpha)^{n+r} + (n+\alpha)\alpha^{2}s\, (x+y+(n+r)\alpha+t\alpha+s\alpha)^{n+r-1}\}.$$

and also we have

$$(1.19)\quad S(1,n+r-1,x+\alpha,1-x) = \int_{o}^{\infty} e^{-t} (1+(n+r)\alpha+t\alpha)^{n+r-1}\, dt.$$

$$(1.20)\quad S(1,n+r-2,x+\alpha,1-x+\alpha) = \int_0^\infty e^{-t}(1+ (n+r)\alpha+t\alpha)^{n+r-2},$$

$$(1.21)\quad S(2,n+r-2,x+2\alpha,1-x) = \int_0^\infty e^{-t}dt \int_0^\infty e^{-s}ds \,\{(x+2\alpha)(1+ (n+r)\alpha+t\alpha+s\alpha)^{n-2}+ (n+r-2)\alpha^2 s\,(1+ (n+r)\alpha+t\alpha+s\alpha)^{n-3}\},$$

$$(1.22)\quad S(2,n+r-3,x+2\alpha,1-x+\alpha) = \int_0^\infty e^{-t}dt \int_0^\infty e^{-s}ds \,\{(x+2\alpha)(1+ (n+r)\alpha+t\alpha+s\alpha)^{n-3}+ (n+r-3)\alpha^2 s\,(1+ (n+r)x+t\alpha+s\alpha)^{n-4}\}.$$

If the function f(x) is defined on the interval (0,b),b>0, the Bernstein polynomial (1.1) for this interval is given by Choldovsky [2] as

$$(1.23)\quad B^f_n(x;b) = \sum_{k=0}^{n} f\left(\frac{bk}{n}\right) (n_k) \left(\frac{x}{b}\right)^k \left(1-\frac{x}{b}\right)^{n-k}.$$

In analogy with above polynomials we set $y = x\,b^{-1}$ in the polynomial $A^{\alpha}_{nr}(\varnothing, y)$ of the function $\varnothing(y)=f(by)$, $0 \leq y \leq 1+\frac{r}{n}$ to obtain the desired generation of the Bernstein polynomials in the following way:

$$(1.24)\quad A^{\alpha}_{nr}(f,x;b) = (n+r+1) \sum_{k=0}^{n+r} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(tb)\,dt\right) p_{nr,k}\left(\frac{x}{b};\alpha\right)$$

where

$$(1.25)\quad P_{nr,k}\left(\frac{x}{b};\alpha\right) = \binom{n+r}{k} \frac{\left(\frac{x}{b}\right)\left(\frac{x}{b}+k\alpha\right)^{k-1}\left(1-\frac{x}{b}+(n+r+k)\,\alpha\right)^{n+r-k}}{(1+(n+r)\,\alpha)^{n+r}}$$

and moreover

$$(1.26) \qquad \sum_{k=o}^{n+r} p_{nr,k}\left(\frac{x}{b};\alpha\right) = 1.$$

For a constant b, we do not obtain any thing essentially now, We shall assume here that $b = b_{nr}$ is a function of (n+r), which increases to $+\infty$ with (n+r), and f(x) is defined in the infinite interval $0 \leq x < +\infty$. If we want to preserve the relation:

$$A^{\alpha}_{nr}(x;b) \longrightarrow f(x).$$

For resonably general class of function f, we have to assumed that the distance between two adjacent point K $\frac{b_{nr}}{n+r}$ tends to zero for $(n+r) \longrightarrow +\infty$, that is, $b_{nr} = 0\,(n+r)$.

Choldovsky [2] has proved the following theorems by assuming that $b = b_n$ is a function of n, which increases to $+\infty$ with n and f(x) is defined in the infinite interval $0 \leq X < +\infty$.

Theorem (1.1): If $b_n = o(n)$ and the function f(x) is bounded in $(0, +\infty)$, say $|f(x)| \leq M$, then $B_n(f,x) \longrightarrow f(x)$ at any point of continuity of the function f.

Theorem (1.2): If $b_n = o(n)$ and $M(b_n)\, e^{-\beta n/b_n} \longrightarrow 0$. for each $\beta > 0$, then $B_n(f;x) \longrightarrow f(x)$ holds at each point of continuity of the function. The aim of this paper is to generalize the above results of Chlodovsky [2] for our new generalized polynomial for Lebesgue integrable function. We prove the following:

Theorem (1.3): If $b_{nr} = o\,(n+r)$, and the function f(x) is bounded Lebesgue integrable in $(0, +\infty)$ say $|f(x)| \leq M$, then for

$$\alpha = \alpha_{nr} = o\left(\frac{1}{n+r}\right),$$

$$A^{\alpha}_{nr}(f,x;b) \longrightarrow f(x)$$

at any point of continuity of the function f.

Theorem (1.4): If $b_{nr} = o\,(n+r)$ and $M(b_{nr})\, e^{-\beta nr/b_{nr}} \longrightarrow 0$

for each $\beta > 0$, then for $\alpha = \alpha_{nr} = o\left(\frac{1}{n+r}\right)$,

$$A^{\alpha}_{nr}(f,x;b) \longrightarrow f(x)$$

at each point of continuity of integrable function f.

2. LEMMAS AND THEIR PROOFS

Lemma (2.1): $\sum_{k=o}^{n+r} k\, p_{nr,k}(x;\alpha) = 0\left(\frac{(n+r)\,x}{1+\alpha}\right).$

Lemma (2.2): $\sum_{k=o}^{n+r} K(k-1)\, p_{nr,k}(x;\alpha) = 0\,(n+r)\,(n+r-1)\alpha\, x$

$$x \left\{ \frac{x+2\alpha}{(1+2\alpha)^2} + \frac{(n+r-2)\,\alpha^2}{(1+3\alpha)^3} \right\}.$$

Lemma (2.3): For all values of $x \in \left[0,1+\frac{r}{n}\right]$ and for $\alpha = \alpha_{nr}$

$= o\left(\frac{1}{n+r}\right)$, we have

$$(n+r+1) \sum_{k=o}^{n+r} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (t-x)^2 \,.\, dt \right) p_{nr,k}(x;\alpha) \leq \frac{x\,(1-x)}{n+r}.$$

Lemma (2.4): For all values of $x \in \left[[0,1+\frac{r}{b}\right]$ and for $\alpha = \alpha_{nr}$

$= o\left[\frac{1}{n+r}\right]$, the inequality

$$0 \leq z \leq \tfrac{3}{2}\,[(n+r) \times (1-x)]^{1/2}$$

implies

$$(n+r+1) \quad \Sigma \quad \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt \right) p_{nr,k}(x;\alpha) \leq 2e^{-z^2}.$$

$$|t-x| > 2z \left(\frac{x(1-x)}{n+r} \right)^{1/2}$$

Proof of Lemma 2.1.: We have

$$\sum_{k=0}^{n+r} k \; p_{nr,k}(x;\alpha) = \sum_{k=0}^{n+r} k \binom{n+r}{k} \frac{x(x+k\alpha)^{k-1}(1-x+(n+r-k)\alpha)^{n+r-k}}{(1+(n+r)\alpha)^{n+r}}$$

$$= \frac{(n+r)\;x}{(1+(n+r)\alpha)^{n+r}} \sum_{k=1}^{n+r} \binom{n+r-1}{k-1} (x+k\alpha)^{k-1} (1-x+(n+r-k)\alpha)^{n+r-k}$$

$$= \frac{(n+r)x}{(1+(n+r)\alpha)^{n+r}} S(1, n+r-1, x+\alpha, 1-x)$$

[by (1.12)]

Using (1.17), we get

$$= \frac{(n+r)x}{(1+(n+r)\alpha)^{n+r}} \int_0^\infty e^{-t}(1+(n+r)\alpha+t\alpha)^{n+r} dt.$$

$$= \frac{(n+r)x}{(1+(n+r)\alpha)^{n+r}} \int_0^\infty e^{-t} \left(1+\frac{t\alpha}{(1+(n+r)\alpha}\right)^{n+r-1} dt\,(1+(n+r)\alpha)^{n+r-1}$$

$$\leq \frac{(n+r)x}{\alpha} \int_0^\infty e^{-}\left(\frac{\alpha}{1}+r+n\right)^{u} . \, e^{(n+r-1)n}\, du,$$

$$= \frac{(n+r)x}{\alpha} \int_0^\infty e^{-}\left(\frac{1}{\alpha}+1\right)^{u} du,$$

and hence

$$\sum_{k=0}^{n+r} k \; p_{nr,k}(x;\alpha) + \frac{(n+r)x}{(1+\alpha)},$$

which completes the proof of lemma (2.1).

Proof of Lemma 2.2: We have

$$\sum_{k=o}^{n+r} k(k-1)\, p_{nr,k}(x;\alpha)$$

$$= \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r}} \sum_{k=o}^{n+r} \binom{n+r-2}{k-2} (x+k\alpha)^{k-1} (1-x+(n+r-k)\alpha)^{n+r-k},$$

$$= \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{nr}} \sum_{v=o}^{n+r-2} \binom{n+r-2}{v} (x+v\alpha+2\alpha)^{v+1} (1-x+(n+r-v-2)\alpha)^{n+r-v-2}$$

$$= \frac{(n+r)(n+r-1)}{(1+(n+r)\alpha)^{n+r}} S(2, n+r-2, x+2\alpha, 1-x), \text{ (by (1.12)]}$$

Using (1.18), we get

$$= \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r}} \Big[\int_o^\infty e^{-t}dt \int_o^\infty e^{-s}\, ds \;\{(x+2\alpha)(1+(n+r)\alpha + t\alpha+s\alpha)^{n+r-2} + (n+r-2)\alpha^2 s (1+(n+r)\alpha + t\alpha+s\alpha)^{n+r-3}\} \Big]$$

(2.1) $= I_1 + I_2$ (say).

First we evaluate I_1,

$$I_1 = \frac{(n+r)(n+r-1)}{(1+(n+r)\alpha)^2} x(x+2\alpha) \int_o^\infty e^{-t}dt \int_o^\infty e^{-s} \left(1 + \frac{(t\alpha+s\alpha)}{1+(n+r)\alpha}\right)^{n+r-2} ds,$$

$$\leq \frac{(n+r)(n+r-1)}{(1+(n+r)\alpha)^2} x(x+2\alpha) \int_o^\infty e^{-t}dt \int_o^\infty e^{s}\, (n+r-2)e$$

$$\left(\frac{t\alpha + s\alpha}{1+(n+r)\alpha}\right) ds$$

$$= \frac{(n+r)(n+r-1)}{(1+(n+r)\alpha)^2} x(x+2\alpha) \int_0^\infty e^{-t'} \left(\frac{1+(n+r)\alpha}{1+2\alpha}\right)) ds' .$$

$$\int_0^\infty e^{-s'} \left(\frac{1+(n+r)\alpha}{1+2\alpha}\right)) ds',$$

$$= \frac{(n+r)(n+r-1)}{(1+2\alpha)^2} x(x+2\alpha) \int_0^\infty e^{-t'} dt \int_0^\infty e^{-s'} ds',$$

and therefore

$$(2.2)\quad I_1 \leq \frac{(n+r)(n+r-1)}{(1+2\alpha)^2} x(x+2\alpha).$$

Now we calculate I_2,

$$I_2 = \frac{(n+r)(n+r-1)(n+r-2)x\,\alpha^2}{(1+(n+r)\alpha)^{n+r}} \int_0^\infty e^{-t}dt \,.\int_0^\infty se^{-s}(1+(n+r)$$

$$\alpha+t\alpha+s\alpha)^{n+r-3}ds,$$

$$\leq \frac{(n+r)(n+r-1)(n+r-2)}{(1+(n+r)\alpha^2} x\,\alpha^2 \int_0^\infty e^{-t}dt \,.$$

$$\int_0^\infty se^{-s} \quad e\,(n+r-3)\left(\frac{t\alpha+s\alpha}{1+(n+r)\alpha}\right) . ds$$

$$= \frac{(n+r)(n+r-1)(n+r-2)}{(1+(n+r)\alpha^3} x\,\alpha^2 \int_0^\infty e^{-t}\left(\frac{1+3\alpha}{1+(n+r)\alpha}\right) dt.$$

$$.\int_0^\infty se^{-s}\left(\frac{1+3\alpha}{1+(n+r)\alpha}\right) ds,$$

$$= \frac{(n+r)(n+r-1)(n+r-2)}{(1+3\alpha)^3} x\,\alpha^2 \int_0^\infty e^{-t'}dt' \int_0^\infty s'\,e^{-s'}\,ds',$$

and hence

$$(2.3)\qquad I_2 \leq \frac{(n+r)(n+r-1)(n+r-2)}{(1+3\alpha)^3} x\,\alpha^2$$

and therefore (2.1), (2.2) and (2.3) gives

$$\sum_{k=0}^{(n+r)} k(k-1)\,p_{nr,k}(x;\alpha) \leq (n+r)(n+r-1)\,x\left(\frac{x+2\alpha}{(1+2\alpha)^2}\right) + \frac{(n+r-2)\,\alpha^2}{(1+3\alpha)^2}$$

which completes the proof of lemma (2.2).

Proof of Lemma (2.3): We have

$$(n+r+1)\sum_{k=0}^{(n+r)}\left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (t-x)^2\,dt\right) p_{nr,k}(x;\alpha)$$

$$= \sum_{k=0}^{(n+r)}\left(x^2 - \frac{2kx+x}{n+r+1} + \frac{k^2+k}{(n+r+1)^2} + \frac{1}{3(n+r+1)^2}\right) p_{nr,k}(x;\alpha)$$

$$\leq x^2 - \frac{1}{(n+r+1)} - \left(\frac{2(n+r)x^2}{1+\alpha} + x\right) +$$

$$+ \frac{1}{(n+r+1)^2}\left[(n+r)(n+r-1)\,x\left\{\frac{x+2\alpha}{(1+2\alpha)^2} + \frac{(n+r-2)\,\alpha^2}{(1+3^{\,3}}\right\}\right.$$

$$\left. + \frac{2(n+r)x}{1+\alpha}\right] + \frac{1}{3(n+r+1)^2},\ \text{(by Lemma 2.1 and 2.2)}$$

$$\leq \frac{1}{(n+r)(1+\alpha)(1+2\alpha)^2(1+3\alpha)^3}\,[x(1-x) + \alpha\,\{x(1-x)(2(n+r)+9) + x\}$$

$$+ \alpha^2 \{x(1-x)(17(n+r)+23)+9x\}+$$
$$+ \alpha^3 \{x(1-x)(57(n+r)-13)+7(n+r)x^2+x(5(n+r)^2+35\}$$
$$+ \alpha^4 \{x(1-x)(96(n+r)-144)+86(n+r)x^2+x(65-12(n+r))\}$$
$$+ \alpha^5 \{x(1-x)(54(n+r)-216)+162(n+r)x^2+x(4(n+r)^2-12(n+r)+46\}$$

$$+ \alpha^3 \{-108\,x(1-x)+108(n+r)x^2\}]+\frac{1}{3(n+r)^2}$$

$$\leq \frac{x(1-x)}{(n+r)},\ \text{for } \alpha=\alpha_{nr}=o\left(\frac{1}{n+r}\right) \text{ and large } (n+r).$$

Which completes the proof of the lemma 2.3.

Proof of Lemma (2.4):

Let Φ be the generating function of the polynomial

$$T=\sum_{k=o}^{(n+r)} (k-(n+r)x)\, p_{nr,k}(x;\alpha)$$

which may be defined as

$$\Phi=\Phi_{nr}(u,s)=\sum_{s=o}^{\infty}\frac{1}{s!}T_{nr,r}(x)\,u^s,$$

$$=\sum_{k=o}^{(n+r)} p_{nr,k}(x;\alpha)\sum_{s=o}^{\infty}\frac{1}{s!}(k-(n+r)x)^s u^s,$$

tet

$$=\sum_{k=o}^{(n+r)} e^{u(k(n+r)x)}\binom{n+r}{k}\cdot\frac{x(x+k\alpha)^{k-1}(1-x(n+r-k)\alpha)^{n+r-k}}{(1+(n+r)\alpha)^{n+r}}$$

$$=e^{-(n+r)xu}\quad(1-x+xe^u)^{n+r},\ \text{for } \alpha=\alpha_{nr}=o\left(\frac{1}{n+r}\right),$$

and therefore

$$(2.4\quad \Phi=[e^{-xu}(1-x+xe^u)]^{(n+r)}$$

To prove our result we will first show that for $|u|\leq 3/2$ the inequality

(2.5) $\Phi \leq \exp\,[(n+r)\,x\,(1-x)\,u^2]$ holds.

For, (2.4) can be written as

$$\Phi = [x\,e^{u(1-x)} + (1-x)\,e^{ux}]^{n+r},$$

but since

$$xe^{u(1-u)} + (1-x)\,e^{-ux} = \sum_{v=o}^{\infty} \frac{u^v}{v!}\,[x(1-x) + (1-x)\,(-x)^v],$$

$$= 1 + \sum_{v=2}^{\infty} \frac{u^v}{v!}\,[x\,(1-x) + (1-x)\,(-x)^v],$$

$$\leq 1 + x(1-x) \sum_{v=2}^{\infty} \frac{|u|^v}{v!},$$

$$\leq 1 + x(1-x)\,\frac{u^2}{2}\,1\left(+\frac{|u|}{3} + \frac{|u|^2}{3^2} + \cdots\right)$$

$$\leq 1 + x(1-x)\,u^2, \text{ for } |u| \leq 3/2$$

$$\leq e^{x(1-x)}u^2, \text{ as } e^k > k+1,$$

and hence

$$\Phi \leq [e^{v(1-x)}u^2]^{n+r}$$

$$= e\,[(n+r)\,x\,(1-x)\,u^2].$$

Therefore if

$$(2.6)\ \Psi = \Psi_{nr}(u,x) = \sum_{k=o}^{(n+r)} e^{u|k-(n+r)x|}\,p_{nr,k}(xi\varkappa)$$

then we obtain for $0 \leq u \leq 3/2$,

$$\Psi \leq \varnothing_{nr}(u,x) + \varnothing_{nr}(-u,x),$$

and, therefore, for $\alpha = \alpha_{nr} = o\left(\frac{1}{n+r}\right)$, we have

(2.7) $\Psi \leq 2\,e^{(n+r)}\,x\,(1-x)\,u^2$

Now to get our required result we note that for $c \geq 0$ and $u \geq o$,

$$(n+r+1) \sum_{\exp[u|k-(n+r)x|] \leq c\Psi} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt \right) p_{nr,k}(x;\alpha) \leq$$

$$\leq \frac{1}{\Psi c}(n+r+1) \sum_{k=o}^{(n+r)} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt \right) e^{u|k-(n+r)x|} p_{nr,k}(x;\alpha)$$

$$\leq \frac{1}{c}.$$

Now if we put $c = \frac{1}{2} z^2$, we obtain

$$(n+r+1) \sum_{\exp[u|k-(n+r)x|] > \frac{1}{2} e^{z^2} \Psi} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt \right) p_{nr,k}(x;\alpha) \leq 2 e^{-z^2},$$

or $$(n+r+1) \sum_{|k-(n+r)x| \leq z^2u^{-1} + (n+r)x(1-x)u} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt \right) p_{nr,k}(x;\alpha) \leq 2 e^{-z^2}.$$

Since for the given range of t, $\left| \frac{k}{n+r} - x \right| \sim |t-x|$,

we have

$$(n+r+1) \sum_{|t-x| \leq z^2 u^{-1}(n+r)^{-1} + x(1-x)u} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt \right)$$

(2.8) $$p_{nr,k}(x;\alpha) \leq 2 e^{-z^2}.$$

Since $0 \leq z \leq \frac{3}{2} [(n+r)x(1-x)]^{1/2}$ can be written as

$$0 \leq z [(n+r)x(1-x)]^{-1/2} \leq 3/2,$$

but (2.8) holds for $0 \leq u \leq 3/2$, and therefore for $u = z((n+r)(1-x)\frac{1}{2}x)$

$$(n+r+1) \sum_{|t-x| \geq z \left(\frac{x(1-x)}{n+r}\right)^{1/2}} + z \left(\frac{x(1-x)}{n+r}\right)^{1/2}$$

$$\left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt\right) p_{nr,k}(x;\alpha) \leq 2\, e^{-z^2},$$

$$(n+r+1) \sum_{|t-x| \geq 2z\left(\frac{x(1-x)}{n+r}\right)^{1/2}} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt\right) p_{nr,k}(x;\alpha)$$

$$\leq 2\, e^{-z^2}.$$

Which completes the proof of Lemma (2.4).

3. PROOF OF THE THEOREMS.

Proof of the Theorem (1.3):. We have

$$| A^{\alpha}_{nr}(f,x;b) - f(x) | \leq$$

$$\leq (n+r+1) \sum_{k=0}^{(n+r)} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} | f(b_{nr}t - f(x) | \, dt\right) p_{nr'k}\left(\frac{x}{b_{nr}};\alpha\right)$$

Let $\epsilon > 0$, be arbitrary and choose $\delta > 0$, so small that

$| f(x) - f(x_1) | < \epsilon$ for $| x - x_1 | < \delta$, we have

$$| A^{\alpha}_{nr}(f,x;b) - f(x) | \leq$$

$$\leq (n+r+1) \sum_{| b_{nr} t - x | < \delta} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} | f(b_{nr}t - f(x) | \, dt\right) p_{nr,k}\left(\frac{x}{b_{nr}};\alpha\right)$$

$$+ (n+r+1) \sum_{| b_{nr} t - x | \geq \delta} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} | f(b_{nr}t - f(x) | \, dt\right) p_{nr,k}\left(\left(\frac{x}{b_{nr}}, \alpha\right)\right.$$

(3.1) $= I_1 + I_2$ (say)

Now

$$I_1 = (n+r+1) \sum_{| b_{nr}t - x | < \delta} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} | f(n_{nr}t) - f(x) | \, dt\right) p_{nr,k}$$

$$\left(\frac{x}{b_{nr}}\ ;\alpha\right)$$

$$< \varepsilon\ (n+r+1) \sum_{k=o}^{(n+r)} \left(\int_{k/(n+r+1)}^{(k+1)/(k+r+1)} dt \right) p_{nr,k} \left(\frac{x}{b_{nr}}\ ;\alpha\right)$$

(3.2) $= \varepsilon$.

Now to evaluate the value of I_2, we put $v = x/b_{nr}$ and we get

$$I_2 = (n+r+1) \sum_{|t-v| \geq \delta/b_{nr}} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |f(v) - f(x)|\, dt \right) p_{nr,k}\,(v;\alpha)$$

$$\leq 2M\,(n+r+1) \sum_{|t-v| \geq \delta/b_{nr}} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt) \right) p_{nr,k}\,(v;\alpha)$$

$$\leq 2M \left(\frac{\delta}{b_{nr}}\right)^{-2} (n+r+1) \sum_{k=o}^{(n+r)} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (t-v)^2 dt \right) p_{nr,k}\,(v;\alpha)$$

$$\leq 2M \left(\frac{\delta}{b_{nr}}\right)^{-2} \left[\frac{v(1-v)}{n+r}\right] \qquad \text{[by lemma (2.3)]}$$

$$\leq 2M \left(\frac{\delta}{b_{nr}}\right)^{-2} \frac{x/b_{nr}}{(n+r)}\ ,$$

(3.3) $\leq \varepsilon$, for large $(n+r)$, since $b_{nr} = o\ (n+r)$,

and hence (3.1), (3.2) and (3.3 gives

$$|A^{\alpha}_{nr}\,(f,\,x;\,b) - f(x)\ | \leq \varepsilon + \varepsilon = 2\,\varepsilon,$$

which completes the proof of theorem (1.3).

Proof of the Theorem (1.4):

Proceeding in a similar manner as in the proof of the theorem (1.3), we obtain (3.1) and taking M (b_{nr}) in place of M, we have

$$|\,A^{\alpha}_{nr}\,(f,x;b) - f(x)\ | \leq (n+r+1) \left(\sum_{|b_{nr}t-x| < \delta} \right) + (n+r+1) \left(\sum_{|b_{nr}\,t-x\ | \geq \delta} \right)$$

$$\leq \epsilon + 2M\,(b_{nr})\,(n+r+1) \sum_{|t-v| \geq \delta/b_{nr}} \left(\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt \right) p_{nr,k}(vx;\alpha)$$

The second term can easily be estimated by means of the lemma (2.4), if

$$z = \delta\,(n+r)\,(2b_{nr})^{-1}\,[\,(n+r)\,v(1-v)\,]^{-1/2},$$

the condition $o \leq z \leq 3/2\,[\,(n+r)\,x\,(1-x)\,]^{1/2}$ is satisfied if we assume, for instance, that $\delta < 2x$ and that $(n+r)$ is sufficiently large

Hence by $M(b_{nr})\,e^{\beta nr}/b^{\beta nr} \longrightarrow 0$, we have

$$|\,A^{\alpha}_{nr}\,(f,x;b) - f(x)\,| \leq \varepsilon + 2M\,(b_{nr})\,e^{-z^2},$$

$$= \varepsilon + 2M\,(b_{nr})\exp\,\{-\,\delta^2(n+r)\,[4\,b_{nr}\,x(1-x\,b^{-1}{}_{nr})\,]^{-1}\},$$

$\leq \varepsilon + \varepsilon = 2\varepsilon$, for all large $(n+r)$,

Which completes the proof of theorem (1.4).

REFERENCES

[1] **Bernstein, S.:** Démonstration du Theoréme de Wierstrass fondeé sur le calcul des probabilitiés, Commun. Soc. Math. Kharkow (2) 13 (1912-13) (1-2).

[2] **Chlodovsky, I:** Sur le devéloppement des functions defines dans un intervl inafinien series de polynomes de M.S. Bernstein. Compositio Maths 4 (1937), 380-393.

[3] **Jenson, J.L.W.V.:** Sur Une indeutite d'Abel at sur d'autres formules analogues. Acta Maths. 26 (1902), 307-318.

[4] **Kantovovic, L.A.:** Sur certain developpements suivant les polynomes de la forme deS-Bernstein I, II. C.R. Acad. Sci URSS (1930), 563-568, 595-500.

[5] **Shurer, F.:** Linear positive operations in approximation Math. Inst. Univ. Tech. Delf. Report 1962.