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REFLECTION PRINCIPLES FOR GENERALIZED POLY-AXIALLY SYMMETRIC BIHARMONIC FUNCTIONS

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ABSTRACT

Some reflection principles for the solutions of a class of fourth order elliptic differential equations with singular coefficients are obtained.

INTRODUCTION

In this article we will consider the elliptic differential operator

$$\triangle_\Sigma := \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x^2_i} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right)$$

where k_i are real constants. The function u is called Σ–biharmonic [Çelebi (1968)] in a region E of the n-dimensional space, if $u \in C^4$ (E) and satisfies the partial differential equation

$$\triangle_\Sigma (\triangle_\Sigma U) = \triangle^2_\Sigma U = 0$$

Similarly u is called Σ–polyharmonic, if $u \in C^{2p}$ (E) and satisfies

$$\triangle_\Sigma (\triangle^{p-1}_\Sigma U) = \triangle^p_\Sigma U = 0 \qquad (1)$$

In the following, we will give some representation formulas for Σ–polyharmonic functions of order p and will obtain a reflection principle for Σ–biharmonic functions.

REPRODUCING Σ–POLYHARMONIC FUNCTIONS

Let us point out two properties that the operator $\triangle_\Sigma$ has. The first one is

$$\triangle^p_\Sigma \triangle^q_{x_i} U = \triangle^q_{x_i} \triangle^p_\Sigma U \tag{2}$$

where

$$\triangle_{x_i} := \frac{\partial^2}{\partial x^2_i} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} .$$

The equation (2) can be obtained easily by an induction from

$$\triangle_\Sigma \triangle_{x_i} U = \triangle_{x_i} \triangle_\Sigma U.$$

The second property of the operator $\triangle_\Sigma$ is

LEMMA 1. Let $u \in C^{2p+1}(E)$. Then, for $p \in N$

$$\triangle^p_\Sigma x_i \frac{\partial}{\partial x_i} U = x_i \frac{\partial}{\partial x_i} \triangle^p_\Sigma U + 2p \triangle_{xi} \triangle^{p-1}_\Sigma U \tag{3}$$

PROOF: We will make use of induction in proving this lemma. By a direct calculation we obtain

$$\triangle_\Sigma x_i \frac{\partial}{\partial x_i} U = x_i \frac{\partial}{\partial x_i} \triangle_\Sigma U + 2 \triangle_{x_i} U \tag{4}$$

Now let us assume that (3) holds for $p = \alpha$:

$$\triangle^\alpha_\Sigma x_i \frac{\partial}{\partial x_i} U = x_i \frac{\partial}{\partial x_i} \triangle^\alpha_\Sigma U + 2 \alpha \triangle_{xi} \triangle^{\alpha-1}_\Sigma U \tag{5}$$

Applying the operator $\triangle_\Sigma$ to both sides of (5) we get

$$\triangle^{\alpha+1}_\Sigma x_i \frac{\partial}{\partial x_i} U = \triangle_\Sigma x_i \frac{\partial}{\partial x_i} \triangle^\alpha_\Sigma U + 2 \alpha \triangle_\Sigma \triangle_{xi} \triangle^{\alpha-1}_\Sigma U. \tag{6}$$

On the other hand, if we replace u by $\triangle^\alpha{}_\Sigma U$ in (4) we obtain

$$\triangle_\Sigma x_i \frac{\partial}{\partial x_i} \triangle^\alpha_\Sigma U = x_i \frac{\partial}{\partial x_i} \triangle^{\alpha+1}_\Sigma U + 2 \triangle_{xi} \triangle^\alpha_\Sigma U. \tag{7}$$

To complete the proof we should substitute (7) in (6):

$$\triangle^{\alpha+1}_\Sigma x_i \frac{\partial}{\partial x_i} U = x_i \frac{\partial}{\partial x_i} \triangle^{\alpha+1}_\Sigma U + 2 (\alpha + 1) \triangle_{x_i} \triangle^\alpha_\Sigma U$$

Now we can state the result on reproducing Σ–polyharmonic functions of order p from a given Σ-polyharmonic function of the same

order. In the following we will denote a solution of (1) by u_p $\{k_1,\ldots,k_n\}$. $\mathcal{U}_p$ (E) will symbolize the set of all solutions of (1), in the domain E.

LEMMA 2. Let u_p $\{k_1,\ldots,k_n\} \in \mathcal{U}_p$ (E) be given. The function defined by

$$x_i^{p+s} \triangle_\Sigma^s \frac{U_p}{x_i^{p-s}}$$

is also a Σ–polyharmonic function of order p, for $x_i \neq 0$, $p > s$ and $p, s \in N$.

PROOF: It is easy to varify for a function $w \in C^{2p}$ (E) that

$$x^3_i \triangle_\Sigma \frac{w}{x_i} = (2 - k_i)\, w + x_i^2 \triangle_\Sigma w - 2\, x_i \frac{\partial w}{\partial x_i} \tag{8}$$

Now, let us apply the operator $\triangle_\Sigma^2$ to both sides of (8), and use Lemma 1.

$$\triangle_\Sigma^2 \left(x_i^3 \triangle_\Sigma \frac{w}{x_i}\right) = (2\text{–}k_i) \triangle_\Sigma^2 w + \triangle_\Sigma^2 (x_i^2 \triangle_\Sigma w) - 2 \triangle_\Sigma^2\, x_i \frac{\partial w}{\partial x_i}$$

$$= \left[3(2+k_i) + x_i^2 \triangle_\Sigma + 6\, x_i \frac{\partial}{\partial x_i}\right] \triangle_\Sigma^2 w \tag{9}$$

From (9), we see that if $w \in \mathcal{U}_2$ (E), then

$$x^3_i \triangle_\Sigma \frac{w}{x_i} \in \mathcal{U}_2 \text{ (E)}.$$

This proves the lemma in the case of $s = 1$, $p = 2$.

Let us assume that

$$x_i^p \triangle_\Sigma \frac{w}{x_i^{p-2}} = (p-2)(p-1-k_i)\, w + x_i^2 \triangle_\Sigma w - 2\,(p-2)\, x_i \frac{\partial w}{\partial x_i}$$

holds. Then,

$$x_i^{p+1} \triangle_\Sigma \frac{w}{x_i^{p-1}} = x_i^{p+1} \triangle_\Sigma \frac{1}{x_i^{p-2}} \left(\frac{w}{x_i}\right)$$

$$= (p-2)\ (p-1-k_i)\, w + x_i^3 \triangle_\Sigma \frac{w}{x_i}$$

$$-\, 2\ (p-2)\ x_i \ \frac{\partial w}{\partial x_i} + 2\,(p-2)\, w$$

is obtained. Using (8) we get

$$x_i^{p+1} \triangle_\Sigma \frac{w}{x_i^{p-1}} = (p-1)\,(p-k_i)\, w + x_i^2 \triangle_\Sigma w - 2\,(p-1)\, x_i \frac{\partial w}{\partial x_i} \tag{10}$$

Now we can show that

$$x_i^{p+1} \triangle_\Sigma \frac{w}{x_i^{p-1}} \in \mathcal{U}_p\,(E)$$

by applying the operator $\triangle_\Sigma^p$ to both sides of (10):

$$\triangle_\Sigma^p \left[x_i^{p+1} \triangle_\Sigma \frac{w}{x_i^{p-1}}\right] = (p-1)\ (p-k_i)\ \triangle_\Sigma^p\, w + \triangle_\Sigma^p\, (x_i^2 \triangle_\Sigma w)$$

$$-\, 2\ (p-1)\ \triangle_\Sigma^p \left(x_i \ \frac{\partial w}{\partial x_i}\right)$$

The property given by Lemma 1 yields

$$\triangle_\Sigma^p \left[x_i^{p+1} \triangle_\Sigma \frac{w}{x_i^{p-1}}\right] = (p-1)\ (p-k_i)\ \triangle_\Sigma\ w$$

$$-\, 2\ (p-1)\ \left[2\,p\ \triangle_{x_i} \triangle_\Sigma^{p-1}\ w + x_i \frac{\partial}{\partial x_i}\ \triangle_\Sigma^p\, w\right]$$

$$+\ \triangle_\Sigma^{p-1} \left[2\ (1+k_i)\ \triangle_\Sigma w\ + 4\ x_i \frac{\partial}{\partial x_i}\ \triangle_\Sigma w + x^2{}_i\ \triangle_\Sigma^2\, w\right].$$

For the sake of simplicity we will assume $w \in \mathcal{U}_p$ (E). Then, by a recursive calculation, we get

$$\triangle_\Sigma^p \left[x_i^{p+1} \triangle_\Sigma \frac{w}{x_i^{p-1}}\right] = -4p\ (p-1)\ \triangle_{x_i}\ \triangle_\Sigma^{p-\alpha}\ w$$

$$+ 4 \triangle_{\Sigma}^{p-1} x_i \frac{\partial}{\partial x_i} \triangle_{\Sigma} w + \triangle_{\Sigma}^{p-1} x_i^2 \triangle_{\Sigma}^{2} w$$

$$= - 4p(p-1) \triangle_{x_i} \triangle_{\Sigma}^{p-1} w + 8(p-1) \triangle_{x_i} \triangle_{\Sigma}^{p-1} w$$

$$+ 8 (p-2) \triangle_{xi} \triangle_{\Sigma}^{p-1} w + \triangle_{\Sigma}^{p-2} x_i^2 \triangle_{\Sigma}^{3} w$$

and finally

$$= [-4p(p-1) + 8 \{(p-1) + (p-2)$$

$$+ \ldots + 1\}] \triangle_{xi} \triangle_{\Sigma}^{p-1} w$$

$$= 0 .$$

That is, if $w \in \mathcal{U}_p (E)$ then there exists a $u_p \{k_1,\ldots,k_n\} \in \mathcal{U}_p (E)$, such that

$$x_i^{p+1} \triangle_{\Sigma} \frac{w}{x_i^{p-1}} = u_p \{k_1,\ldots,k_n\}$$

To complete the proof of the lemma by induction, we will show that

$$x_i^{p+s} \triangle_{\Sigma}^{s} \left(\frac{w}{x_i^{p-s}}\right)$$

is a Σ–polyharmonic function of order p, for $s = 2$. First of all, notice that

$$x_i^{p+2} \triangle_{\Sigma}^{2} \frac{w}{x_i^{p-2}} = x_i^{p+2} \triangle_{\Sigma} \left(\frac{v}{x_i^{p}}\right)$$

where

$$v = x_i^{p} \triangle_{\Sigma} \frac{w}{x_i^{p-2}}$$

The function v is a Σ–polyharmonic function of order $p - 1$. Obviously $v \in \mathcal{U}_p (E)$. So there exists a function $u_p \{k_1,\ldots,k_n\} \in \mathcal{U}_p (E)$ such that

$$x_i^{p+2} \triangle_{\Sigma} \left(\frac{v}{x_i^{p}}\right) = x_i^{p+2} \triangle_{\Sigma}^{2} \left(\frac{w}{x_i^{p-2}}\right) = u_p \{k_1,\ldots,k_n\}.$$

For the last step of the proof, assume that there exists a $u_p \{k_1, \ldots,k_n\} \in \mathcal{U}_p (E)$ such that

$$x_i^{p+\nu} \triangle_{\Sigma}^{\nu} \left(\frac{w}{x_i^{p-\nu}}\right) = u_p \{k_1,\ldots,k_n\} \tag{11}$$

for a given $\nu \in N$. Then we can write

$$x_i^{p+\nu+1} \mathop{\triangle}_{\Sigma}^{\nu+1} \left(\frac{w}{x_i^{p-\nu-1}} \right) = x_i^{p+\nu+1} \triangle_{\Sigma} \left\{ \left(x_i^{p+\nu-1} \mathop{\triangle}_{\Sigma}^{\nu} \frac{w}{x_i^{p-\nu-1}} \right) \Big/ x_i^{p+\nu-1} \right\} \tag{12}$$

Using (11), we obtain

$$x_i^{p+\nu-1} \mathop{\triangle}_{\Sigma}^{\nu} \frac{w}{x_i^{p-\nu-1}} \in \mathcal{U}_{p-1} (E)$$

and

$$x_i^{p+\nu-1} \mathop{\triangle}_{\Sigma}^{\nu} \frac{w}{x_i^{p-\nu-1}} \in \mathcal{U}_p (E)$$

So, from (12) we reach the result that there exists a function $u_p \{k_1, \ldots, k_n\} \in \mathcal{U}_p (E)$, such that

$$x_i^{p+\nu+1} \mathop{\triangle}_{\Sigma}^{\nu+1} \frac{w}{x_i^{p-\nu-1}} = U_p \{k_1, \ldots, k_n\}$$

for $p - \nu - 1 > 0$, if $w \in \mathcal{U}_p (E)$.

REMARK: It can easily be shown that Lemma 2 holds for $p \leq s$ by using the results obtained elsewhere [Çelebi (1968); Süray and Çelebi (1973)].

A REFLECTION PRINCIPLE IN THE CASE OF TWO INDEPENDENT VARIABLES

We will introduce the following notations:

$$H = \{P : x > 0\},$$
$$D = \{P : x = 0\},$$
$$C_r = \{P : x^2 + y^2 < r^2\}$$

where $P \in \mathbb{R}^2$ and $r \in \mathbb{R}^+$. $\overline{H}$ and $\overline{C}_r$ are the closures of H and C_r, respectively.

THEOREM 1: Let $u_2 \{k_1, k_2\} \in \mathcal{U}_2 (C_r \cap H)$. If the function $u_2 \{k_1, k_2\}$ satisfies the condition

a) $\lim_{x \to 0} x^{k_1} u_2 \{k_1, k_2\} = 0$, for $k_1 > 0$

or

b) $\lim_{x \to 0} u_2 \{k_1, k_2\} = 0$, for $k_1 < 0$

in the domain $S \subset D$, then $u_2 \{k_1, k_2\}$ can be continued to the region $C_r \cap (\sim \overline{H})$ in the form

$$u^*_2(-x,y) = -u_2(x,y) + 4(6-k_1-k_2)\, x^{1-k_1} y^{1-k_2} + 4(5-k_1)\, x^{3-k_1} y^{1-k_2} + 4(5-k_2)\, x^{1-k_1} y^{3-k_2} + \triangle_\Sigma \{(-x)^{1-k_1} y^{1-k_2} [x^4 + y^4 + x^2y^2 + x^2 + y^2]\} \quad (12)$$

as a Σ–biharmonic function, where $\sim \overline{H}$ is the complement of $\overline{H}$.

PROOF: In order to prove the theorem [Rabadi (1983)], we have to show that the following statements hold:

i) The function $u^*_2(-x,y)$ defined by (12) is Σ–biharmonic;

ii) If $(x,y) \in C_r \cap (\sim \overline{H})$ then

$$\begin{aligned} u_2(-x,y) = & -u^*_2(x,y) + 4(6-k_1-k_2)(-x)^{1-k1} y^{1-k2} \\ & + 4(5-k_1)(-x)^{3-k1} y^{1-k2} + 4(5-k_2)(-x)^{1-k1} y^{3-k2} \\ & + \triangle_\Sigma \{x^{1-k1} y^{1-k2} [x^4 + y^4 + x^2y^2 + x^2 + y^2]\}; \end{aligned}$$

iii) The function

$$U(x,y) = \begin{cases} u_2(x,y), & (x,y) \in C_r \cap H \\ u_2(x,y), & (x,y) \in C_r \cap (\sim \overline{H}) \end{cases}$$

is continuous on D.

The first statement follows from a direct computation. If we apply the operator $\triangle^2_\Sigma$ to both sides of (12), we obtain

$$\triangle^2_\Sigma u^*_2 = 0$$

which implies that $u^*_2 \in \mathcal{U}_2 [C_r \cap (\sim \overline{H})]$.

The second statement is trivial.

For the last statement we should prove that

a) $\lim_{x \to 0} x^{k_1} u^*_2 = 0$, for $k_1 > 0$

or

b) $\lim_{x \to 0} u^*_2 = 0$, for $k_1 < 0$.

But this is evident from (12).

Thus the function u^*_2 defined by (12) is a continuous extension of the function u_2 to the region $C_r \cap (\sim \overline{H})$. Moreover, we easily can obtain that x^{k_1} U (x,y) is analytic in a domain not containing the x-axis, if $x^{k_1} u_2(x,y)$ is analytic.

A REFLECTION PRINCIPLE FOR Σ-BIHARMONIC FUNCTIONS

First of all, we will introduce some more notations in R^n, similar to that of the section above

$$H_i = \{P : x_i > 0\},$$
$$D_i = \{P : x_i = 0\},$$
$$C_r = \left\{ P : \sum_{i=1}^{n} x_i^2 < r^2 \right\},$$
$$C_{r,i} = C_r \cap H_i$$

where $P \in R^n$ and $\overline{H}_i$, $\overline{D}_i$, $\overline{C}_r$, $\overline{C}_{r,i}$ are the closures of H_i, D_i, C_r, $C_{r,i}$, respectively.

THEOREM 2. Let $u_2 \{k_1,\ldots,k_n\} \in \mathcal{U}_2 (C_{r,i})$, and let $k_i \neq 0$ for a given i. If the function $u_2 \{k_1,\ldots,k_n)$ satisfies the condition

a) $\lim_{P \to P_o} x_i^{1-k_i} u_2 \{k_1,\ldots,k_n\} = 0$ for $k_i > 0$

or

b) $\lim_{P \to P_o} x_i^{k_i - 1} u_2 \{k_1,\ldots,k_n\} = 0$ for $k_i < 0$

for $P_o \in S \subset D_i$, then $u_2 \{k_1,\ldots,k_n\}$ can be extended continuously to the domain $C_r \cap (\sim \overline{H}_i)$ in the form of

$$u^*_2(P^*) = \frac{1}{k_i - 1} \left\{ u_2(P) - x^3_i \triangle_\Sigma \frac{u_2(P)}{x_i} \right\} \tag{13}$$

as a Σ–biharmonic function where P^* is the reflection of P with respect to D_i, and $\sim \overline{H}_i$ is the complement of $\overline{H}_i$.

PROOF: To prove the theorem, we must show that the following statements hold:

i) $u^*_2(P^*)$, defined by (13) is Σ–biharmonic;

ii) $$u_2(P) = \frac{1}{k_i - 1} \left[u^*_2(P^*) - x^3_i \triangle_\Sigma \frac{u_2^*(P^*)}{x_i} \right] \tag{14}$$

iii) The function

$$U(x_1, \ldots, x_n) = \begin{cases} u_2(P) & ; \ P \in C_{r,i} \\ u^*_2(P^*) & ; \ P^* \in C_r \cap (\sim \overline{H}_i) \end{cases}$$

is continuous on D_i.

For the proof of the first statement, we can write

$$\left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{\partial^2}{\partial x^2_j} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) + \frac{\partial^2}{\partial(-x_j)^2} + \frac{k_i}{(-x_i)} \frac{\partial}{\partial(-x_i)} \right]^{(2)} u^*_2(P^*)$$

$$= \triangle^2_\Sigma \left\{ \frac{1}{(k_i - 1)} \left[u_2(P) - x^3_i \triangle_\Sigma \frac{u_2(P)}{x_i} \right] \right\}$$

$$= \frac{1}{k_i - 1} \left[\triangle^2_\Sigma u_2(P) - \triangle^2_\Sigma \left\{ x_i^3 \triangle_\Sigma \frac{u_2(P)}{x_i} \right\} \right]$$

and by Lemma 2

$$\triangle^2_\Sigma u^*_2(P^*) = 0.$$

So, the function $u^*_2(P^*)$, defined by (13) is Σ–biharmonic.

The second statement is a consequence of a direct calculation. First we will substitute (13) in the right hand side of (14):

$$\frac{1}{k_i - 1} \left[u^*_2(P^*) - x_i^3 \triangle_\Sigma \frac{u^*_2(P^*)}{x_i} \right]$$

$$= \frac{1}{(k_i-1)^2}\left[u_2(P) - 2\,x_i^3\triangle_\Sigma \frac{u_2(P)}{x_i} + x_i^3\triangle_\Sigma\left((x_i^2\triangle_\Sigma \frac{u_2(P)}{x_i}\right)\right]$$

$$= \frac{1}{(k_i-1)^2}\left[u_2(P) - 2\,x_i^3\triangle_\Sigma \frac{u_2(P)}{x_i} + (2-k_i)\,x_i^3\triangle_\Sigma \frac{u_2(P)}{x_i}\right.$$

$$\left. + x_i^3\triangle_\Sigma\,(x_i\triangle_\Sigma u_2(P)\,) - 2\,x^3{}_i\triangle_\Sigma \frac{\partial\, u_2(P)}{\partial\, x_i}\right]$$

$$= \frac{1}{(k_i-1)^2}\left[u_2(P) - k_i x_i^3\left\{(2-k_i)\,x_i^{-3}u_2(P) + \frac{1}{x_i}\triangle_\Sigma u_2(P)\right.\right.$$

$$\left.- \frac{2}{x_i^2}\frac{\partial\, u_2(P)}{\partial\, x_i}\right\} + x^3{}_i\left\{\frac{k_i}{x_i}\triangle_\Sigma u_2(P) + x_i\triangle_\Sigma^2 u_2(P)\right.$$

$$\left.\left.+ 2\frac{\partial}{\partial\, x_i}\triangle_\Sigma u_2(P)\right\} - 2\,x_i^3\triangle_\Sigma \frac{\partial\, u_2(P)}{\partial\, x_i}\right]$$

$$= \frac{1}{(k_i-1)^2}\left[(k_i-1)^2\,u_2(P) + 2\,k_i x_i \frac{\partial\, u_2(P)}{\partial\, x_i}\right.$$

$$\left.+ 2\,x^3{}_i \frac{\partial}{\partial\, x_i}\triangle_\Sigma u_2\,(P) - 2\,x_i^3\,\triangle_\Sigma \frac{\partial\, u_2\,(P)}{\partial\, x_i}\right]$$

$$= u_2\,(P) + \frac{1}{(k_i-i)^2}\left[2\,k_i x_i \frac{\partial}{\partial\, x_i}\triangle_\Sigma u_2\,(P)\right.$$

$$\left.-2\,x_i^3\left(\triangle_\Sigma \frac{\partial\, u_2\,(P)}{\partial\, x_i} - \frac{\partial}{\partial\, x_i}\triangle_\Sigma u_2(P)\right)\right]$$

$$= u_2(P).$$

This shows that the second statement holds.

In order to obtain the third statement, we will assume $P \in C_r \cap (\sim \overline{H}_i)$ and $P_o \in S$. Then, it is easy to verify that either

$$\lim_{P\to P_o} x_i^{1-k_i}\, u^*{}_2\,\{k_1,\ldots,k_n\} = 0 \qquad \text{for } k_i > 0$$

or

$$\lim_{P \to P_o} x_i^{k_i - 1} u^*_2 \{k_1, \ldots, k_n\} = 0 \qquad \text{for } k_i < 0.$$

Thus, we have established the reflection principle for Σ–biharmonic functions with respect to a singular hypersurface.

REMARK: The special case

$$k_i = 2 \text{ and } k_j = 0, \; j \neq i, \; j = 1, \ldots, n$$

of the above theorem gives us the results obtained in Armitage (1978), Duffin (1955) and Rabadi (1983) for the harmonic functions.

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