COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1: Mathématiques

TOME : 33 ANNÉE : 1984

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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

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Constancy of Holomorphic Sectional Curvature in Pseudo-Kähler Manifolds

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(Received March 2, 1984 and accepted August 13, 1984)

ABSTRACT

Cartan [1] had proved that a Riemannian Manifold is of constant curvature if R(X,Y, X,Z) = 0 for every orthonormal triplet X,Y and Z. Graves and Nomizu [2] have extended this result to Pseudo-Riemannian Manifold. In the present paper this result has been extended to Kahler Manifolds with indefinite metric by proving that: "A Pseudo-Kahler manifold (M, J) is of constant Holomorphic Sectional Curvature if R(X,Y,X,JX) = 0 whenever X,Y and JX are orthonormal". A result of Tannö [4] on Almost Hermitian Manifold has also been extended to Pseudo-Kahler Manifolds by proving that a criterian for constancy of Holomorphic Sectional Curvature is that R(X,JX) X is proportional to JX.

1. INTRODUCTION

Definition: A Kähler manifold (M,J) with structure tensor J, endowed with a Pseudo Riemannian metric g shall be called a Pseudo-Kähler manifold.

If X is a vector field on a Pseudo Kähler Manifold M. We shall say that

$$X \text{ is space like if } \quad g(X,X) > 0,$$
$$X \text{ is time like if } \quad g(X,X) < 0,$$
$$X \text{ is null if } \quad g(X,X) = 0, \qquad X \neq 0.$$

The metric is said to be degenerate if $\exists\, X \in \chi(M) \ni g(X,Y) = 0$ $\forall\, Y \in \chi(M)$.

A submanifold N of M shall be called non-degenerate (degenerate) if the restriction of g to N is non-degenerate (degenerate).

First, we establish the following lemma which will be useful in our discussion.

Lemma (1.1) [2]. The plane $p = sp\{X,Y\}$ is non-degenerate if and only if

$$g(X,X)\ g(Y,Y) - g(X,Y)\ g(X,Y) \neq 0.$$

Proof: Let us consider a fixed vector $\xi = \alpha X + \beta Y$ and an arbitrary vector $Z = xX + yY$, both in the plane p. Then p is not-degenerate iff $g(Z,\xi) = 0\ \forall\ Z$ implies $\xi = 0$.

Now

$0 = g(\xi,Z) = \{\alpha g(X,X) + \beta g(X,Y)\}\ x + \{\alpha g(X,Y) + \beta g(Y,Y)\}\ y$ where x and y are arbitrary.

This implies

$$\alpha g(X,X) + \beta g(X,Y) = 0 \qquad \text{(a)}$$

and

$$\alpha g(X,Y) + \beta g(Y,Y) = 0 \qquad \text{(b)}$$

These equations would admit $\alpha = 0$, $\beta = 0$ as the only solution if and only if

$$g(X,Y)\ g(X,Y) - g(X,Y)\ g(X,Y) \neq 0.$$

Q.E.D.

Corollary (1.2): The plane $p = sp\{X,JX\}$ is non-degenerate if and only if $g(X,X) \neq 0$.

2. CURVATURE TENSOR

It is easily seen that the following properties hold for the curvature tensor R of a Pseudo-Kähler manifold as well

$$R(X,Y)\ J = J\ R(X,Y)$$

and (1)

$$R(JX,\ JY) = R(X,Y)$$

for all vector fields X and Y.

A plane section p is called holomorphic if $Jp = p$. The holomorphic sectional curvature K(p) of such a plane section is equal to R(X, JX, X, JX). It is well known that a Kähler manifold is of constant holomorphic sectional curvature if and only if

$$R(X,Y)Z = (C/4)\ [g(X,Z)Y - g(Y,Z)X + g(Y,JZ)JX - g(X,JZ)JY - 2g(X,JY)JZ] \tag{2}$$

Definition In a Pseudo-Kähler manifold M, a plane shall be called J-invariant if it is spanned by $\{X, JX\}$.

We shall also use the following result (see e.g. [3]).

Proposition (2.1): If T is any quadrlinear mapping satisfying all the symmetry properties of the covariant curvature tensor, then

$R(X,JX,X,JX) = T(X,JX,X,JX)$ for all X implies $R = T$.

We now, define a new tensor

$$R_0(X,Y,Z,W) = \frac{1}{4}\{g(X,Z)\ g(Y,W) - g(X,W)\ g(Y,Z) + g(X,JZ)\ g(Y,JW) - g(X,JW)\ g(Y,JZ) + 2g(X,JY)\ g(Z,JW)\} \tag{3}$$

Then it is easy to prove

$$R_0(X,JX,X,JX) = g(X,X)^2.$$

Lemma (2.2): All non-degenerate J-invariant planes have same holomorphic sectional curvature, $K(p) = c$, if and only if

$$R = cR_0.$$

Proof: If $R = cR_0$, then obviously $K(p) = c$. Conversely, if $K(p) = c$, then consider the following cases:

(i) Let $g(X,X) \neq 0$ and $p = sp\ \{X,JX\}$. Then

$$R(X,JX,X,JX) = cR_0\ (X,JX,X,JX) \tag{4}$$

(ii) Let now $g(X,X) = 0$. We can always find a sequence of non null vectors $\{X_n\}$ such that $X_n \to X$ in the sense that $g(X_n - X, X_n - X)^2 < \varepsilon$ for $n \geq n_0$ and $g(X_n, X_n) \neq 0$. Then the planes $\{X_n, JX_n\}$ are non-degenerate and hence

$$(R - cR_0)\ (X_n,JX_n,X_n,JX_n) = 0 \quad \text{for all } n.$$

This implies

$$R(X,JX,X,JX) = cR_0(X,JX,X,JX) \tag{5}$$

Thus (4) and (5) show that

$$R(X,JX,X,JX) = cR_0(X,JX,X,JX) \text{ for all } X.$$

Hence by proposition (2.1) $R = cR_0$

Q.E.D.

3. EXTENSION OF CARTAN'S LEMMA FOR PSEUDO-KÄHLER MANIFOLDS

Graves and Nomizu [2] have proved the following result which was proved by Cartan [1] for the positive definite metric.

Theorem (3.1): Let M be a (Pseudo) Riemannian Manifold with indefinite metric g. If $R(X,Y,X,Z) = 0$ for orthonormal vectors X,Y and Z, then all non-degenerate planes have the same sectional curvature.

We shall extend this result for Pseudo-Kähler Manifolds by proving the following:

Theorem (3.2): Let (M,J) be a Pseudo-Kähler Manifold with real dimension ≥ 6. If $R(X,Y,X,JX) = 0$ for every orthonormal set of vectors X,Y and JX, then M is of constant holomorphic sectional curvature.

To prove the above theorem we first establish the following lemma:

Lemma (3.3): The hypothesis of the theorem implies that

$$K\ \{sp(X,JX)\} = K\ \{sp(Y,JY)\}.$$

Proof: Case I: Let $g(X,X) = g(Y,Y) = 1$. Define X' and Y' by

$$X' = \frac{X+Y}{\sqrt{2}} \text{ and } Y' = \frac{JX-JY}{\sqrt{2}}$$

Then X', Y', JX' from an orthonormal set. By the hypothesis of the theorem, we have

$$0 = R(X',Y',X',JX')$$

$$= \frac{1}{4} R(X+Y,\ JX-JY,\ X+Y,\ JX+JY) \tag{6}$$

Using the fact that $R(X,JX,JX,Y) = 0$, we can get directly from (6) that

$$0 = R(X,JX,X,JX) - R(Y,JY,Y,JY)$$

that is,

$$K\ \{sp(X,JX)\} = K\ \{sp(Y,JY)\}$$

Case II: If $g(X,X) = -g(Y,Y)$ Let us define

$X' = aX + bY$ and $Y' = bJX + aJY$, where a and b are two numbers such that

$$a^2 - b^2 = 1.$$

Then X', Y' and JX' form an orthonormal set. Thus we have

$$0 = R(X',Y',X',JX')$$

From which it is easy to prove that

$R(X,JX,X,JX) = R(Y,JY,Y,JY)$. This implies that

$K\ \{sp(X,JX)\} = K\ \{sp(Y,JY)\}$.

Now, if sp $\{U,V\}$ is holomorphic i.e. sp $\{U,V\}$ = sp $\{JU, JV\}$ Then $JU = aU + bV$ i.e. sp $\{U,JU\}$ = sp $\{U,aU + bV\}$ = sp $\{U,V\}$ Similarly, sp $\{V,JV\}$ = sp $\{U,V\}$ i.e. sp $\{U,JU\}$ = sp $\{V,JV\}$ or, $K\ \{sp(U,JU)\} = K\ \{sp(V,JV)\}$

If $\{U,V\}$ is not holomorphic section, then we can choose unit vectors $X \varepsilon \{U,JU\}^{\perp}$ and $y \varepsilon \{V,JV\}^{\perp}$ which determine a holomorphic section $\{X,Y\}$. This implies, as above.

$$K\ \{sp(X,JX)\} = K\ \{sp(Y,JY)\}. \tag{7}$$

$X \in \{U,JU\}^{\perp}$ so $\{U,JU,X\}$ is orthonormal. This implies

$$R(U,X,U,JU) = 0$$

i.e. $K\ \{sp(X,JX)\} = K\ \{sp(U,JU)\}$ (8)

Similarly, $K\ \{sp(Y,JY)\} = K\ \{sp(V,JV)\}$ (9)

from (7), (8) and (9) we have,

$$K\ \{sp(U,JU)\} = K\ \{sp(X,JX)\} = K\ \{sp(Y,JY)\} = K\ \{sp(V,JV)\}.$$

That is, any holomorphic section has same sectional curvature.

Hence by Schur's Theorem, M is of constant holomorphic sectional curvature.

Q.E.D.

4. ANOTHER CRITERIAN FOR CONSTANCY OF HOLOMORPHIC SECTIONAL CURVATURE

Let (M, g, J) be an almost Hermitian manifold of dim ≥ 4. Then $g(JX,JY) = g(X,Y)$ and $J^2X = - X$. Assume that M has the following property:

$$R(JX,JY,JX,JZ) = R(X,Y,X,Z) \qquad (10)$$

for every tangent vectors X,Y and Z. For this class of manifolds, Tanno [4] has proved the following:

"Let $m \geq 4$. Assume that an almost Hermitian manifold satisfies (10). Then it is of constant holomorphic sectional curvature at x if and only if R(X,JX) X is proportional to JX for every tangent vector X at x".

It is well known that every Kähler manifold satisfies (10). In this section we extend the result of Tanno [4] for the class of Pseudo-Kahler manifolds, which obviously satisfies (10). We have.

Theorem (4.1): Let (M,J) be a Pseudo-Kähler manifold of dim ≥ 4. Then M is of constant holomorphic sectional curvature if and only if R(X,JX) is proportional to JX for every tangent vector X.

Proof: If $R(X,JX)X = cJX$ then it follows obviously from (1) and (2) that $K(X,JX) = c$ for all X. To prove the converse we shall consider the following cases:

i) $g(X,X) = g(Y,Y)$ and

ii) $g(X,X) = - g(Y,Y)$

Let {X,Y,JX) be an orthonormal set and assume $m \geq 6$. Define X′ and Z′ by $X' = \frac{X+Y}{\sqrt{2}}$ and $Z' = \frac{JX-JY}{\sqrt{2}}$. Then X′, JX′ and Z′ form an orthonommal set. By hypothesis of the theorem, we have

$$0 = R(X',JX',X',Z')$$

Therefore, from Lemma (3.3) we get $H(X) = H(Y)$, where

$$H(X) = K \{sp(X,JX)\}.$$

Next, we assume $m = 4$. For the first case viz. $g(X,X) = g(Y,Y)$ the theorem is proved in [4]. So we consider the case when $g(X,X) = 1$ and $g(Y,Y) = -1$. In this case we get

$$\begin{aligned}
R(X,JX)X &= H(X)\, JX,\\
R(X,JX)Y &= -R(X,JX,Y,JY)\, JY,\\
R(X,JY)X &= -R(X,JY,X,Y)\, Y - R(X,JY,X,JY)\, JY,\\
R(X,JY)Y &= R(X,JY,Y,X)\, X + R(X,JY,Y,JX)\, JX,\\
R(Y,JY)X &= R(Y,JY,X,JX)\, JX,\\
R(Y,JX)Y &= R(Y,JX,Y,X)\, X + R(Y,JX,Y,JX)\, JX,\\
R(Y,JX)X &= -R(Y,JX,X,Y)\, Y - R(Y,JX,X,JY)\, JY,\\
R(Y,JY)Y &= -H(Y)\, JY = -H(X)\, JY
\end{aligned}$$

Now define $X' = aX + bY$ with $a^2 - b^2 = 1$, then using the above relations we get

$$R(X',JX')X' = C_1X + C_2Y + C_3JX + C_4JY$$

where C_1 and C_2 are separately zero and

$$\begin{aligned}
C_3 &= a^3\, H(X) + ab^2\, C_5 \qquad (11)\\
C_4 &= -b^3\, H(X) - a^2b\, C_5, \quad \text{where}
\end{aligned}$$

$$C_5 = R(X,JX,Y,JY) + R(X,JY,Y,JX) + R(Y,JX,Y,JX).$$

On the other hand

$$\begin{aligned}
R(X',JX')X' &= H(X')\, JX'\\
&= H(X')\, (aJX + bJY) \qquad (12)
\end{aligned}$$

Comparing (11) and (12) we get

$$a^2\, H(X) + b^2\, C_5 = H(X')$$

and

$$-b^2\, H(X) - a^2\, C_5 = H(X')$$

From last two equations we get $C_5 = -H(X)$

Thus $H(X') = (a^2 - b^2)\, H(X) = H(X)$

Similarly we can prove $H(Y') = H(Y)$, and thus M has a constant holomorphic sectional curvature.

Q.E.D.

Thus we have shown that both these criteria (viz. [2] and [4] can be extended to Pseudo-Kähler manifolds.

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