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CONVOLUTIONS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ $b_n \geq 0$. We investigate some properties of $h(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$ where f (z) and g (z) satisfy either $\mathrm{Re}(f(z)/z) > \alpha$, $\mathrm{Re}(g(z)/z) > \alpha$ or $\mathrm{Re}\, f'(z) > \alpha$, $\mathrm{Re}\, g'(z) > \alpha$ for $|z| < 1$.

INTRODUCTION

Let S denote the class of functions normalized by $f(0) = f'(0) - 1 = 0$ that are analytic and univalent in the unit disk E. A function $f(z) \in S$ is said to be starlike if $\mathrm{Re}\,(zf'(z)/f(z)) > 0$ for $|z|<1$ and is said to be convex if $\mathrm{Re}\left(1 + \frac{zf''(z))}{f'(z)}\right) > 0$ for $|z| < 1$. These classes are denoted by S^* and K respectively.

The convolution or Hadmard product of two power series

$$f(z) = \sum_{n=o}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=o}^{\infty} b_n z^n$$

is defined as the power series $(f * g)(z) = \sum_{n=o}^{\infty} a_n b_n z^n$.

Ruscheweyh and T. Sheil-small (1973) proved the Polya-Schoenberg conjecture that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in K,$$

then

$$h(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \in K.$$

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geqq 0$ and let $P(\alpha)$ denote the class of functions of the form $f(z)$ which satisfy $\mathrm{Re}(f(z)/z) > \alpha$ for $|z| < 1$ and $Q(\alpha)$ denote the class of functions $f(z)$ which satisfy $\mathrm{Re}\, f'(z) > \alpha$ for $|z| < 1$. In this paper we obtain some properties of $h(z) = f(z) * g(z)$ where $f(z)$ and $g(z)$ belongto $P(\alpha)$ or $Q(\alpha)$ for $0 \leq \alpha < 1$.

Schild and Silverman (1975) investigated some properties of convolutions of univalent functions with negative coefficients.

CONVOLUTION PROPERTIES

We need the following result:

LEMMA:

i. $f(z) \in P(\alpha)$ iff $\sum_{n=2}^{\infty} a_n \leq 1 - \alpha$

ii. $f(z) \in Q(\alpha)$ iff $\sum_{n=2}^{\infty} n a_n \leq 1 - \alpha$.

PROOF: The lemma has been proved in Sarangi and Uralegaddi (1978)

THEOREM 1: If $f(z) \in P(\alpha)$ and $g(z) \in P(\alpha)$ then $h(z) = f(z) * g(z)$

$$= z - \sum_{n=2}^{\infty} a_n b_n z^n \in P(2\alpha - \alpha^2).$$

PROOF: From Lemma we have

$$\sum_{n=2}^{\infty} a_n \leq 1 - \alpha \text{ and } \sum_{n=2}^{\infty} b_n \leq 1 - \alpha.$$

In view of Lemma, we have to find the largest $\beta = \beta(\alpha)$ such that

$$\sum_{n=2}^{\infty} a_n b_n \leq 1 - \beta.$$

We have to show that

$$\sum_{n=2}^{\infty} \frac{a_n}{1-\alpha} \leq 1 \tag{1}$$

and

$$\sum_{n=2}^{\infty} \frac{b_n}{1-\alpha} \leq 1 \tag{2}$$

imply that

$$\sum_{n=2}^{\infty} \frac{a_n b_n}{1-\beta} \leq 1 \text{ for all } \beta = \beta(\alpha) = 2\alpha - \alpha^2. \tag{3}$$

From (1) and (2) we obtain by means of Cauchy – Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{\sqrt{a_n}\sqrt{b_n}}{1-\alpha} \leq 1 \tag{4}$$

Hence it is sufficient to prove that

$$\frac{a_n b_n}{1-\beta} \leq \frac{\sqrt{a_n}\sqrt{b_n}}{1-\alpha}, \beta = \beta(\alpha), n = 2,3 \ldots \text{ or } \sqrt{a_n}\sqrt{b_n} \leq \frac{1-\beta}{1-\alpha}$$

From (4) we have $\sqrt{a_n}\sqrt{b_n} \leq 1 - \alpha$ for each n. Hence it will be sufficient to show that

$$1 - \alpha \leq \frac{1-\beta}{1-\alpha} \tag{5}$$

Solving for β we get $\beta \leq 2\alpha - \alpha^2$.

The result is sharp with equality for $f(z) = g(z) = z - (1-\alpha) z^2$.

COROLLARY: Let $f(z) \in P(\alpha)$, $g(z) \in P(\alpha)$ and let

$h(z) = z - \sum_{n=2}^{\infty} \sqrt{a_n}\sqrt{b_n}\, z^n$. Then $\operatorname{Re}(h(z)/z) > \alpha$ for $|z| < 1$.

This result follows from the inequality (4). It is sharp for the same functions as in Theorem 1.

THEOREM 2. Let $f(z) \in P(\alpha)$ and $g(z) \in P(\beta)$, then

$$h(z) = f(z) * g(z) \in P(\alpha + \beta - \alpha\beta).$$

PROOF: The proof is similar to that of Theorem 1.

COROLLARY: Let $f(z) \in P(\alpha)$, $g(z) \in P(\beta)$ and $h(z) \in P(\Gamma)$, then

$$f(z) * g(z) * h(z) \in P(\alpha + \beta + \Gamma - \alpha\beta - \beta\Gamma - \Gamma\alpha + \alpha\beta\Gamma).$$

THEOREM 3: Let $f(z) \in Q(\alpha)$ and $g(z) \in Q(\beta)$, then

$$h(z) = f(z) * g(z) \in Q\left(\frac{1 + \alpha + \beta - \alpha\beta}{2}\right).$$

PROOF: From Lemma, we know that

$$\sum_{n=2}^{\infty} \frac{na_n}{1-\alpha} \leq 1 \text{ and}$$

$$\sum_{n=2}^{\infty} \frac{n\, b_n}{1-\beta} \leq 1.$$

We have to find the largest $\Gamma = \Gamma(\alpha, \beta)$ such that

$$\sum_{n=2}^{\infty} n\, a_n\, b_n \leq 1 - \Gamma.$$

It is sufficient to show that $\sum_{n=2}^{\infty} \frac{na_n}{1-\alpha} \leq 1$ and $\sum_{n=2}^{\infty} \frac{nb_n}{1-\beta} \leq 1$

imply $\sum_{n=2}^{\infty} \frac{n\, a_n\, b_n}{1-\Gamma} \leq 1$ for all $\Gamma = \Gamma(\alpha, \beta) = \frac{1+\alpha+\beta-\alpha\beta}{2}$.

Proceeding similarly as in the proof of Theorem 1 we get

$$\frac{a_n b_n}{1-\Gamma} \leq \frac{n\, a_n\, b_n}{(1-\alpha)(1-\beta)} \quad \text{or } \Gamma \leq 1 - \frac{(1-\alpha)(1-\beta)}{n}$$

The right-hand side is an increasing function of n ($n=2,3,\ldots$). Taking $n=2$, we get $\Gamma \leq \frac{1+\alpha+\beta-\alpha\beta}{2}$.

THEOREM 4: Let $f(z) \in Q(\alpha)$ and $g(z) \in Q(\beta)$. Then

$$f(z) * g(z) \in P\left(\frac{3+\alpha+\beta-\alpha\beta}{4}\right).$$

PROOF: From Lemma, we have

$$\sum_{n=2}^{\infty} na_n \leq 1-\alpha \text{ and } \sum_{n=2}^{\infty} nb_n \leq 1-\beta.$$

We have to find the largest $\Gamma = \Gamma(\alpha, \beta)$ such that

$$\sum_{n=2}^{\infty} a_n b_n \leq 1-\Gamma.$$

This is satisfied if

$$\frac{1}{1-\Gamma} \leq \frac{n^2}{(1-\alpha)(1-\beta)} \text{ i.e., for } \Gamma \leq 1-\frac{(1-\alpha)(1-\beta)}{n^2},$$

Since The right-hand side is an increasing function of n, taking $n = 2$ we get the result.

THEOREM 5: If $f, g \in Q(\alpha)$, then

$$h(z) = z - \sum_{n=2}^{\infty} (a_n^2 + b_n^2) z^n \in Q(2\alpha-\alpha^2).$$

PROOF: Since $\sum_{n=2}^{\infty} n a_n \leq 1-\alpha$, we have

$$\sum_{n=2}^{\infty} \frac{n^2 a_n^2}{(1-\alpha)^2} \leq \left(\sum_{n=2}^{\infty} \frac{n a_n}{1-\alpha}\right)^2 \leq 1.$$

Similarly, $\sum_{n=2}^{\infty} \frac{n^2 b_n^2}{(1-\alpha)^2} \leq 1$ and therefore

$$\sum_{n=2}^{\infty} \frac{1}{2} n^2 \frac{a_n^2+b_n^2}{(1-\alpha)^2} \leq 1.$$

We have to find the largest $\beta = \beta(\alpha)$ such that

$$\sum_{n=2}^{\infty} \frac{n}{1-\beta}(a_n^2+b_n^2) \leq 1.$$

This will be satisfied if

$$\frac{1}{1-\beta} \leq \frac{1}{2} \frac{n}{(1-\alpha)^2}$$

or $\beta \leq 1 - \dfrac{2(1-\alpha)^2}{n}$.

Again since the right-hand side is an increasing function of n, we get

$$\beta \leq 2\alpha - \alpha^2.$$

NOTE: The result is sharp for the functions

$$f(z) = g(z) = z - \frac{1}{2}(1-\alpha)z^2.$$

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