Commun. Fac. Sci. Univ. Ank. Series A_1
V. 36, pp 23-25 (1987)

CURVATURE OF SPHERES IN MINKOWSKI (n+1)-SPACE

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(Received: 4 Februaruy 1987. Accepted: 20 May 1987)

ABSTRACT

In this paper we show that tbe hypersurfaces S (α) in Minkowski (n+1) - space M^{n+1} given by:

$$x_1^2 + \ldots + x^2_n + \alpha = x^2_{n+1}, \qquad \alpha \neq 0,$$

carry induced Riemannian metrics which is positive nefinite in case $\alpha > 0$ and negative definite in case $\alpha < 0$. Also the curvature of S (α) is constant and equal to $-n\,(n-1)/\alpha$ for $\alpha > 0$ and $n\,(n-1)/\alpha$ for $\alpha < 0$.

THE MAIN RESULT

The "positive spheres" S (α) in Minkowski (n+1) - space M^{n+1} is given by:

$$\{(x_1, x_2, \ldots, x_{n+1}) \in R^{n+1} : x_1^2 + \ldots + x^2_n + \alpha = x^2_{n+1}, \alpha \neq 0\}$$

This is a hyperbolid of revolution of two sheets diffeomorphic to $R^n \times S^o$. Thus S (α) is a smooth hypersurface in R^{n+1}. The Lorentz structure on R^{n+1} induces a symmetric bilinear form on the tangent space T_xS to S at each $x \in S$. This form h can be calculated as follows:

Without loss of generality we can restrict attention to the positive spheres S_+ where $x_{n+1} > o$. and $\alpha > o$. There is a coordinate system on S_+ given by projection to the first n coordinates in R^{n+1}, and

$$ds^2 = d\,x_1^2 + d\,x^2_2 + \ldots + d\,x_n^2 - d\,x^2_{n+1}, \text{ but}$$

$$x^2_{n+1} = x^2_1 + x^2_2 + \ldots + x^2_n + \alpha, \text{ and hence}$$

$$dx^2_{n+1} = (x_1\, dx_1 + \ldots + x_n\, dx_n)^2 / \triangle,$$

where

$$\triangle = x^2_1 + \ldots + x^2_n + \alpha.$$

Now ds^2 takes the form

$$ds^2 = \sum_{k=1}^{n} dx^2_k - \frac{1}{\triangle} \sum_{i,j=1}^{n} x_i x_j\, dx_i\, dx_j$$

$$= \sum_{i,j=1}^{n} h_{ij}\, dx_j,$$

thus the matrix H_x representing the form h_x for any $x \in S$ is given by

$$H_x = \begin{bmatrix} 1 - \frac{x^2_1}{\triangle} & - \frac{x_1x_2}{\triangle} & \ldots & - \frac{x_1x_n}{\triangle} \\ - \frac{x_1x_2}{\triangle} & 1 - \frac{x^2_2}{\triangle} & \ldots\ldots & - \frac{x_2x_n}{\triangle} \\ \ldots & \ldots & \ldots & \ldots \\ - \frac{x_nx_1}{\triangle} & - \frac{x_nx_2}{\triangle} & \ldots\ldots & 1 - \frac{x^2_n}{\triangle} \end{bmatrix}$$

It is not difficult to show that $|H_x - \lambda I| = 0$ has n positive eigen values, i.e., the metric h_x has signature n and hence is a positive definite.

Similary in the case $\alpha < 0$, the induced Riemannian metric will be negative definite.

To calculate the curvature we consider first the case $n = 2$. In this case the covariant and contravariant components of the form h are:

$$h_{11} = 1 - \frac{x^2_1}{\triangle}, \quad h_{12} = h_{21} = - \frac{x_1x_2}{\triangle}, \quad h_{22} = 1 - \frac{x^2_2}{\triangle},$$

$$h^{11} = \frac{\alpha + x^2_1}{\alpha}, \quad h^{12} = h^{21} = \frac{x_1x_2}{\alpha}, \quad h^{22} = \frac{\alpha + x^2_2}{\alpha}$$

Also the affine connections Γ^c_{ab} have the components:

$$\Gamma^1_{11} = - \frac{x_1\,(\alpha + x^2_2)}{\alpha \triangle}, \quad \Gamma^2_{11} = - \frac{x_2\,(\alpha + x^2_2)}{\alpha \triangle},$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{x^2_1 x_2}{\alpha \triangle}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{x_1 x^2_2}{\alpha \triangle},$$

$$\Gamma^1{}_{22} = -\frac{x_1(\alpha + x^2{}_1)}{\alpha \triangle}, \quad \Gamma^2{}_{22} = -\frac{x_2(\alpha + x^2{}_1)}{\alpha \triangle}.$$

By using the formula,

$$R_{ab} = \frac{\partial \Gamma^c{}_{ab}}{\partial x^c} - \frac{\partial \Gamma^c{}_{ac}}{\partial x^b} + \Gamma^d{}_{ab}\, \Gamma^c{}_{dc} - \Gamma^d{}_{ac}\, \Gamma^c{}_{db},$$

for the Ricci tensors, it is easy to check that:

$$R_{11} = -\frac{(\alpha + x^2{}_2)}{\alpha \triangle}, \; R_{12} = \frac{x_1 x_2}{\alpha \triangle} = R_{21}, \; R_{22} = -\frac{(\alpha + x^2{}_1)}{\alpha \triangle},$$

and hence the curvature of S_+ will be

$$R = -2/\alpha.$$

Now, for $n = 3$, the Ricci tensors are given by:

$$R_{11} = -\frac{2}{\alpha \triangle}(\alpha + x^2{}_2 + x^2{}_3), \quad R_{12} = R_{21} = \frac{2x_1x_2}{\alpha \triangle}$$

$$R_{22} = -\frac{2}{\alpha \triangle}(\alpha + x^2{}_1 + x^2{}_3), \quad R_{23} = R_{32} = \frac{2\, x_2 x_3}{\alpha \triangle},$$

$$R_{33} = -\frac{2}{\alpha \triangle}(\alpha + x^2{}_1 + x^2{}_2), \quad R_{13} = R_{31} = \frac{2x_1x_3}{\alpha \triangle},$$

and hence $R = -\frac{(3)\,(2)}{\alpha}$. This can be generalized for the n case for the hypersurfaces $S(\alpha)$ where $\alpha > 0$ and once again for $\alpha < 0$ to have

$$R = \begin{cases} n\,(n-1)/\alpha, & n \geqq 2 \\ 1/\alpha, & n = 1, \end{cases}$$

where the curvature R is negative for $\alpha > 0$ and positive for $\alpha < 0$.

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