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ON FOUR DIMENSIONAL EINSTEIN MANIFOLDS

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ABSTRACT

In this paper we provide a study of 4-dimensional Einstein manifolds, in particular we classify all 4-dimensional Einstein manifolds (up to local isometry) which have tbe property that tbe metric depends on only one coordinate.

Kasner (1923) considered the solutions of the Einstein equations involving functions of only one variable. His calculations are rather long and purely algebraic in nature. In this paper, by using a theorem of Singer and Thorpe (1969), we provide an alternative approach to find all 4–dimensional Einstein manifolds (up to local isometry) which have the property that the metric depends on only one coordinate. To do this we consider the metric

$$ds^2 = dt^2 + \sum_{i=1}^{3} l_i^2(t)\ dx_i^2$$

where t is the arc-length and the following orthogonal vector fields:

$$T = \frac{\partial}{\partial t},\ X_1,\ X_2 \text{ and } X_3 \text{ such that } |T| = 1,\ |X_i| = l_i(t),\ i=1,2,3.$$

Then, for the covariant differentiation ∇, we have

$$\nabla_{X_i} X_j . X_k = 0,\ \nabla_{X_i} X_j . T = 0,\ \nabla_T X_i . X_j = 0 \quad (i \neq j)$$

$$\nabla_{X_i} X_i . T = -l_i l'_i,\ \nabla_{X_i} T . X_i = l_i l'_i,$$

so that $\nabla_{X_i} X_i = -l_i l'_i T$, $\nabla_{X_i} T = \nabla_T X_i = \frac{l'_i}{l_i} X_i$, $\nabla_{X_i} X_j = 0$ $(i \neq j)$

and these determine the curvature as follows:

$$R_{X_iX_j}\, X_k = 0 \qquad \text{(i, j, k are distinct)},$$

$$R_{X_iX_j}\, X_i = -\nabla_{X_j}(-\, l_i l'_i T) = \frac{l_i l'_i l_i'}{l_j}\, X_i,\ R_{X_iX_jX_iX_j} = l_i l'_i l_j l'_j,$$

$$R_{X_iX_j}\, T = \nabla_{X_i}\left(\frac{l'_j}{l_j} X_j\right) - \nabla_{X_j}\left(\frac{l'_i}{l_i} X_i\right) = 0,$$

$$R_{X_iT}\, X_i = \nabla_{X_i}\left(\frac{l'_i}{l_i} X_i\right) + \nabla_T l_i l'_i\, T = (-l'^2_i + l_i'^2 + l_i l_i'')\, T = l_i l_i'' T,$$

$$R_{X_iTX_iT} = l_i l_i'',\ R_{X_jT}\, X_j = \nabla_{X_i}\left(\frac{l'_j}{l_j} X_j\right) = 0 \ (i \neq j).$$

Thus the sectional curvatures are

$$\sigma_{ij} = -\frac{R_{X_iX_jX_iX_j}}{|X_i \Lambda X_j|^2} = -\frac{l'_i l'_j}{l_i l_j},\ \sigma_{iT} = -\frac{R_{X_iTX_iT}}{|X_i \Lambda T|^2} = \frac{l''_i}{l_i}, \tag{1}$$

Now by the Theorem 1.4 of Singer and Thorpe (1969), the manifold M with the above metric is Einstein if and only if

$$\frac{l'_i l'_j}{l_i l_j} = \frac{l''_k}{l_k} \qquad \text{(i, j, k are distinct)}. \tag{2}$$

Note that

$$\left(\frac{l'_k}{l_k}\right)' = \frac{l''_k l_k - l_k'^2}{l_k^2} = \frac{l''_k}{l_k} - \left(\frac{l'_k}{l_k}\right)^2$$

This and (2) show that if

$$(u, v, w) = \left(\frac{l'_1}{l_1}, \frac{l'_2}{l_2}, \frac{l'_3}{l_3}\right)$$

then

$$(u, v, w)' = (vw - u^2, uw - v^2, uv - w^2). \tag{3}$$

The system of non-linear differential equations in (3), namely, the system

$$\begin{aligned} u' &= vw - u^2 \\ v' &= uw - v^2 \\ w' &= uv - w^2 \end{aligned} \tag{4}$$

turns out to be much more easily solved than we might expect. This is because geometry tells us that each (u, v, w) which satisfies (3) is tangent to the surface $uv+uw+vw=c$ (c=constant), one reason for this is, since $-uv$, $-uw$ and $-vw$ denote the sectional curvatures, an Einstein manifold M must have scalar curvature $-(uv+uw+vw)$ equal to some constant provided $\dim M \geqq 3$. Therefore the condition $uv+uw+vw=c$ reduces the system (4) from 3 -space to a surface. But further, since $vw=c-uv-uw$ we can write

$$vw - u^2 = c - u(u+v+w),$$

similarly

$$uw - v^2 = c - v(u+v+w),$$

$$uv - w^2 = c - w(u+v+w).$$

Thus

$$(u,v,w)' = (c,c,c) - (u+v+w)(u,v,w),$$

which says that the tangent vector $(u,v,w)'$ is in the plane through (0, 0, 0), (1,1,1) and (u,v,w), and the solution curve is the intersection of this plane P and the hyperboloid $uv+uw+vw=c$.

This argument shows that there is exactly one solution through each point of $\mathbb{R}^4$ and so we can find them. We now state the following

THEOREM: Let M be a 4-dimensional Einstein manifold whose metric depends on only one coordinate, that is, M has the following metric

$$ds^2 = dt^2 + l^2_1(t)dx^2_1 + l^2_2(t)dx^2_2 + l^2_3(t)dx^2_3.$$

Then M is one of the following four types:

i) $l_i(t) = e^{at+c_i}$ (a, c_i constants) i=1,2,3 and thus M is of constant sectional curvature (hence it is locally symmetric).

ii) $l_i(t) = c_i(t+k)^{a_i+1/3}$ where $c_i > 0$, k are constants and $a_1+a_2+a_3=0$, $a_1{}^2+a^2{}_2+a^2{}_3=6$. The sectional curvatures are

$$\sigma_{ij} = -\frac{(a_i+1)(a_j+1)}{9(t+k)^2} \quad i,j=1,2,3,\ i\neq j.$$

For $(a_1,a_2,a_3) = (2,-1,-1)$ (or its permutations) M is flat.

iii) $l_i(t) = c_i (\sin 3t)^{1/3} (\csc 3t - \cot 3t)^{a_i/3}$ where $c_i > 0$ constant and a_i's are as in (ii). The sectional curvatures are

$$\sigma_{ij} = -\frac{1}{s^2}\left[1 - s^2 + (a_i + a_j)\sqrt{1 - s^2} + a_i a_j\right]$$

where $s = \sin 3t$

iv) $l_i(t) = c_i (\sinh 3t)^{1/3} (\tanh 3t/2)^{a_i/3}$ where $c_i > 0$ constant and a_i's are as in (ii). The sectional curvatures are

$$\sigma_{ij} = -\frac{1}{4s^2(1+s^2)}(1 + 2s^2 + a_i)(1 + 2s^2 + a_j)$$

where $s = \sinh(3t/2)$. For $(a_1, a_2, a_3) = (2, -1, -1)$, M is complete and analytic.

Proof: i) If the solution of (4) is constant, i.e. if the solution is on the diagonal of the cone (to which the hyperboloids are tangent), then we have $u' = v' = w' = 0$ and from (4) it follows that $(u, v, w) = (a, a, a)$ (a = constant) is a solution. In this case $l_i = e^{at + c_i}$ (c_i = constant), and thus we obtain manifolds with constant sectional curvature $-a^2$. Since constant sectional curvature implies locally symmetric, these manifolds are locally symmetric.

ii) Note that by (4) and the fact that $uv + uw + vw = c$, we have

$$(u+v+w)^2 = u^2 + v^2 + w^2 + 2(uv + uw + vw) = c - (u' + v' + w') + 2c$$
$$= 3c - (u' + v' + w').$$

Now if $c = 0$, then we obtain $u + v + w = 1/t + k$ (k = constant) and from $u' = c - u(u + v + w)$ we get $u = a/t + k$ where a is constant. To determine a, we consider a vector $V = (a_1, a_2, a_3)$ such that

$$a_1 + a_2 + a_3 = 0$$
$$a^2_1 + a^2_2 + a^2_3 = 6. \tag{5}$$

Then

$$(u, v, w) = \left(\frac{a_1 + 1}{3(t+k)}, \frac{a_2 + 1}{3(t+k)}, \frac{a_3 + 1}{3(t+k)}\right)$$

is a solution of (4) and it is on the cone. In this case $l_i(t) = c_i (t+k)^{(a_i+1)/3}$ where $c_i > 0$ constant, and the sectional curvatures are

$$\sigma_{ij} = -\frac{(a_i+1)(a_j+1)}{9(t+k)^2} \quad i,j=1,2,3 \text{ and } i\neq j.$$

As t approaches -k, the curvatures approach infinity, provided $a_i\neq -1 \neq a_j$. We note that the only integer solutions of (5) are (–2,1,1) (and its permutations) and (2,–1,–1) (and its permutations) in which case σ_{ij} become zero, so we obtain flat manifolds locally isometric to $\mathbb{R}^4$.

iii) If we let $V = (a_1,a_2,a_3)$ be as in (5), then $(u,v,w) = \cot 3t\,(1,1,1) - \csc 3t\, V$ is a solution curve which is outside the cone. In this case we have

$l_i = c_i\,(\sin 3t)^{1/3}(\csc 3t - \cot 3t)^{a_i/3}$ where $c_i > 0$ constant. If we let $s = \sin 3t$, then the sectional curvatures become

$$\sigma_{ij} = -\frac{1}{s^2}\left[1 - s^2 + (a_i+a_j)\sqrt{1-s^2} + a_i a_j\right] \tag{1.6}$$

and as t tends to infinity, s does not approach a number, so the curvatures do not approach a specific number.

iv) Let $V = (a_1,a_2,a_3)$ be as in (5). We note that $(u,v,w) = \coth 3t\,(1,1,1) - \operatorname{csch} 3t\,V$ is also a solution curve which is inside the cone. In this case

$$l_i(t) = c_i\,(\sin 3t)^{1/3}(\tanh 3t/2)^{a_i/3}$$

and the curvatures are

$$\sigma_{ij} = -\frac{1}{4s^2(1+s^2)}(1+2s^2+a_i)(1+2s^2+a_j)$$

where $s = \sinh(3t/2)$. As t tends to zero, s tends to zero and the curvatures approach infinity, provided $a_i\neq -1\neq a_j$. If $a_i=-1=a_j$, that is, in the special case where $(a_1,a_2,a_3) = (2,-1,-1)$, as s tends to zero $\sigma_{12}\to -3/2$, and $\sigma_{23}\to 0$. Moreover as s tends to infinity all curvatures approach –1. Therefore this is the only case where for all critical values of t the sectional curvatures are finite numbers. We will now show that the manifold M with these sectional curvatures is complete and analytic.

For $t > 0$ we consider the mapping of M to $\mathbb{R}^4-\{0\}$ given by

$$(t,x_1,x_2,x_3) \overset{\mu}{\to} (t\cos ax_1,\ t\sin ax_1,\ x_2,\ x_3).$$

We determine a so that we can extend the mapping to $\mathbb{R}^4$ and make it a local isometry.

Observe that if g is a smooth function, then

$$g(y_1, y_2, y_3, y_4) = g(t \cos ax_1, t \sin ax_1, x_2, x_3)$$

and so

$$\begin{aligned}
\frac{\partial g}{\partial t} &= \cos ax_1 \frac{\partial g}{\partial y_1} + \sin ax_1 \frac{\partial g}{\partial y_2} \\
\frac{\partial g}{\partial x_1} &= -\, at \sin ax_1 \frac{\partial g}{\partial y_1} + at \cos ax_1 \frac{\partial g}{\partial y_2} \\
\frac{\partial g}{\partial x_2} &= \frac{\partial g}{\partial y_3} \\
\frac{\partial g}{\partial x_3} &= \frac{\partial g}{\partial y_4}
\end{aligned} \tag{6}$$

From (6) we obtain the following vector fields

$$Y_1 = \cos ax_1 T - \frac{\sin ax_1}{at} X_1$$

$$Y_2 = \sin ax_1 T + \frac{\cos ax_1}{at} X_1$$

$$Y_3 = X_2, \qquad Y_4 = X_3.$$

Note that we have

$$\begin{aligned}
Y_1 . Y_1 &= \cos^2 ax_1 + \left(\frac{l_1}{at}\right)^2 \sin^2 ax_1 \\
Y_1 . Y_2 &= \sin ax_1 . \cos ax_1 - \left(\frac{l_1}{at}\right)^2 \sin ax_1 . \cos ax_1 \\
Y_2 . Y_2 &= \sin^2 ax_1 + \left(\frac{l_1}{at}\right)^2 \cos^2 ax_1
\end{aligned} \tag{7}$$

$$Y_1 . Y_3 = Y_1 . Y_4 = Y_2 . Y_3 = Y_2 . Y_4 = Y_3 . Y_4 = 0.$$

If we choose a, so that

$$\lim_{t \to 0} \frac{l_1}{at} = 1 \tag{8}$$

we then make Y_1, Y_2, Y_3, Y_4 orthogonal at $t = 0$. From (8) it follows that $a = 3 \times 2^{-2/3} c_1$.

Since

$$\sin ax_1 = \frac{y_2}{\sqrt{y_1^2 + y_2^2}}, \quad \cos ax_1 = \frac{y_1}{\sqrt{y_1^2 + y_2^2}},$$

$$l_1 = c_1 2^{1/3} s\,(1 + s^2)^{1/6}$$

and $s = \sinh (3\,t/2)$, (7) becomes

$$\begin{aligned}
Y_1.Y_1 &= \frac{y^2{}_1}{y^2{}_1+y^2{}_2} + \frac{4}{9}\,\frac{y^2{}_2}{(y^2{}_1+y^2{}_2)^2}\,\sinh^2 \frac{3\sqrt{y^2{}_1+y^2{}_2}}{2} \\
&\qquad \cosh^{1/3} \frac{3\sqrt{y^2{}_1+y^2{}_2}}{2} \\
Y_1.Y_2 &= \frac{y_1 y_2}{y^2{}_1+y^2{}_2} - \frac{4}{9}\,\frac{y_1 y_2}{(y^2{}_1+y^2{}_2)^2}\,\sinh^2 \frac{3\sqrt{y^2{}_1+y^2{}_2}}{2} \\
&\qquad \cosh^{1/3} \frac{3\sqrt{y^2{}_1+y^2{}_2}}{2} \qquad (9) \\
Y_2.Y_2 &= \frac{y^2{}_2}{y^2{}_1+y^2{}_2} + \frac{4}{9}\,\frac{y^2{}_1}{(y^2{}_1+y^2{}_2)^2}\,\sinh^2 \frac{3\sqrt{y^2{}_1+y^2{}_2}}{2} \\
&\qquad \cosh^{1/3} \frac{3\sqrt{y^2{}_1+y^2{}_2}}{2}
\end{aligned}$$

If we consider the power series expansions

$$\sinh u = u + \frac{u^3}{6} + \ldots, \quad \cosh u = 1 + \frac{u^2}{1} + \frac{u^4}{24} + \ldots$$

so that

$$\sinh^2 u = u^2 + \frac{u^4}{3} + \ldots, \quad \cosh^{1/3} u = 1 + \frac{u^2}{6} + \ldots$$

the functions in (9) take the form

$$Y_1.Y_1 = \frac{y^2_1}{y^2_1+y^2_2} + \frac{4}{9}\frac{y^2_2}{(y^2_1+y^2_2)^2}\left[\frac{9}{4}(y^2_1+y^2_2) + \frac{27}{16}(y^2_1+y^2_2)^2+\dots\right]\left[1+\frac{3}{8}(y^2_1+y^2_2)+\dots\right]$$

$$Y_1.Y_2 = \frac{y_1y_2}{y^2_1+y^2_2} - \frac{4}{9}\frac{y_1y_2}{(y^2_1+y^2_2)^2}\left[\frac{9}{4}(y^2_1+y^2_2) + \frac{27}{16}(y^2_1+y^2_2)^2+\dots\right]\left[1+\frac{3}{8}(y^2_1+y^2_2)+\dots\right]$$

$$Y_2.Y_2 = \frac{y^2_2}{y^2_1+y^2_2} + \frac{4}{9}\frac{y^2_1}{(y^2_1+y^2_2)^2}\left[\frac{9}{4}(y^2_1+y^2_2) + \frac{27}{16}(y^2_1+y^2_2)^2+\dots\right]\left[1+\frac{3}{8}(y^2_1+y^2_2)+\dots\right] \quad (10)$$

Note that these functions are defined and real analytic in IR^4, therefore the manifold M, with the new metric, is real analytic.

We now show that M is complete. For this we compare the lengths of the coordinate vector fields, that is, compare 1, l_1, l_2, l_3 and 1, at, 1, 1 respectively. Under μ_*, X_1 has the length at. This can be seen from the following 2–dimensional illustration.

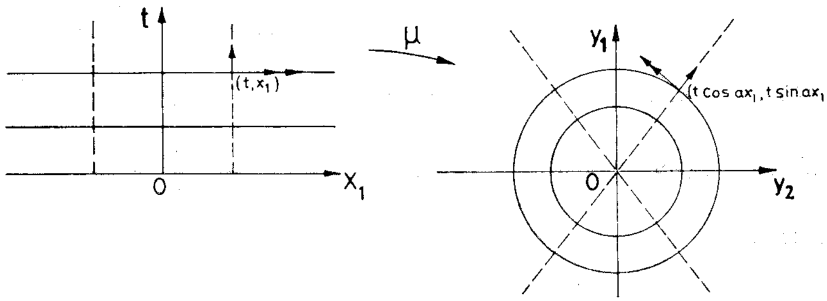

Fig. 1 2 - dimensional illustratin of μ

Note that at is approximately (2 / 3) $as = c_1 2^{1/3} s$ ($s = \sinh 3t/2$, $a = 3 \times 2^{-2/3} c_1$) and so $l_1 = c_1 2^{1/3} s\ (1+s^2)^{1/6} \geqq c'_1$ at, where $c'_1 \geqq 1$ is some constant. Also $l_2 = l_3 = c_2 2^{1/3}\ (1+s^2)^{1/3} \geqq c_2 2^{1/3}$ provided $c_2 \geqq 1$. Now let $c = \min(c_2 2^{1/3}, c'_1)$. Then $c \geqq 1$ and if $V = \Sigma a_i X_i$ is a vector,

$$|V| = \sqrt{\Sigma a_i^2 l_i^2} \geqq \sqrt{\Sigma a_i^2 c^2} = |c| \quad ||V||$$

where $||\,.\,||$ is the euclidean norm of $\mathbb{R}^4$ and $|\,.\,|$ is the norm on M. Therefore the distance funsction on M is bigger than or equal to the euclidean distance in $\mathbb{R}^4$. From this fact it follows that any Cauchy sequence in M is a Cauchy sequence in $\mathbb{R}^4$. Since $\mathbb{R}^4$ is complete, every Cauchy sequence converges. Therefore M is complete.

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