Commun. Fac. Sci. Univ. Ank. Series A_1
V. 37, pp 23–26 (1988)

THE DEFORMATION RETRACT AND TOPOLOGICAL FOLDING OF A MANIFOLD.

M. EL-GHOUL

Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt.

ABSTRACT

In this paper the relation between topological foldings of a manifold M and its deformation retract have been discussed. Also the condition for the two manifolds M, N to have deformation retracts of the same homotopy type has been obtained. And the condition for the converse has been deduced.

INTRODUCTION

A map $F: M \to N$, where M, N are C^∞ Riemannian manifolds of dimensions m, n respectively, is said to be an isometric folding of M into N, iff for any piecewise geodesic path $\gamma : J \to M$, the induced path $F o \gamma: J \to N$ is a piecewise geodesic and of the same length as γ. If F does not preserve length, then F is a topological folding [1], [2].
We discuss the deformation retract of a manifold M, $M = M_1, \cup M_2$, $M_1 \cap M_2 \neq \varnothing$, M_1, M_2 are manifolds, dim M_1 = dim M_2.

DEFINITIONS

1– A subset A of a manifold M is called a deformation retract of M [3] if there exists a retraction $R: M \to A$ and a homotopy $f: M \times I \to M$ such that

$$\left.\begin{array}{l} f(x, 0) = x \\ f(x, 1) = R(x) \end{array}\right\} \forall x \in M$$

$f(a, t) = a$, $a \in A$ and $t \in I = [0, 1]$

2– Two continuous maps $\Phi_0, \Phi_1: M \to N$, M, N are C^∞ Riemannian manifolds, are homotopic if and only if there exists a continuous map $\Phi: M \times I \to N$ such that, for

$x \in M$

$\Phi(x, 0) = \Phi_0(x)$,

$\Phi(x, 1) = \Phi_1(x)$ [3].

THE MAIN RESULTS

THEOREM 1 :

If A is a retraction of an open manifold M, dim A = dim M, $M = A \cup M_1$ and $A \cap M_1 = p \neq \varnothing$ if $f: M \to M$ is a topological folding s. t. f (p) = p, f (A) = A, then f is a deformation retract, the converse is not true, the deformation retract of an open manifold M is not always a topological folding.

Proof:

Let $f: M_1 \to A$ be a topological folding of M_1 into A, fig (1) (a), $M_1 = M' - \{b\}$ where $\{b\}$ is a point of the manifold M_1 to make M_1 open, A

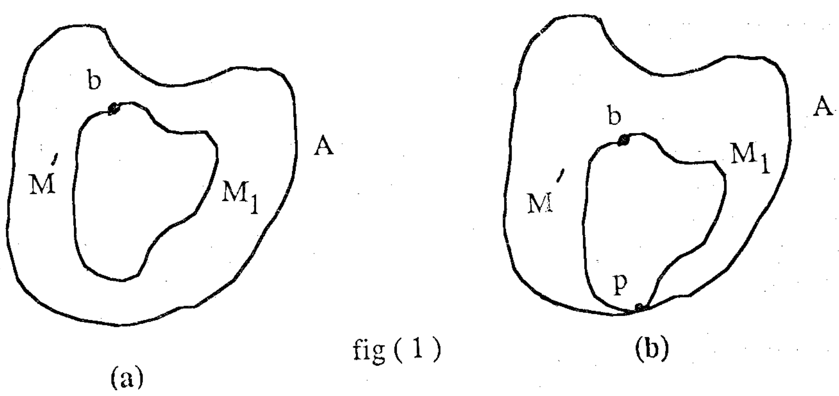

fig (1)

is not a deformation retract of $M = M_1 \cup A$, $M_1 \cap A = \varnothing$, i.e. M is not connected, if $f: M \to M$ is a topological folding, $M = M_1 \cup A$, $M_1 \cap A = \{p\}$, f (p) = p, f (A) = A, then f is a deformation retract of M fig (1) (b), to prove the converse of this result is not true,

take $M = D^n - \{0\} = \{x \in R^n : 0 < |x| < 1\}$,

$A = S^{n-1} = \{x \in R^n : |x| = 1\}$,

take $f: M \times 1 \to M$ s. t.

$f(x, t) = (1 - t)x + t. x / |x|$,

then f is continuous, $f(x, 0) = x$, $f(x, 1) = x/|x| \in S^{n-1}$, and, if $x \in S^{n-1}$ then $f(x, t) = x \; \forall \; t \in I$. This proves that f is a deformation retract, but it is not a topological folding.

THEOREM 2 :

If M and N are two manifolds with the same homotopy type; A is a deformation retract of M and B is a deformation retract of N. Then A and B are of the same homotopy type and the converse is true.

Proof:

Since the two manifolds M and N are of the same homotopy type, then there exists continuous maps $f: M \to N$ and $g: N \to M$ such that $g \circ f = i_M$ and $f \circ g = i_N$, where i_M, i_N are identity maps on M and N respectively. Since A is a deformation retract of M, then there exists a retraction

$r_1 : M \to A$ and a homotopy

$f_1 : M \times I \to M$ s. t.

$$\left.\begin{aligned} f_1(x, 0) &= x \\ f_1(x, 1) &= r_1(x) \end{aligned}\right\} \quad \forall \; x \in M$$

$f_1(a, t) = a$, $a \in A$ and $t \in I = [0, 1]$,

and in the same manner B is a deformation retract of N then there exists $r_2: N \to B$

and $f_2: N \times I \to N$ s. t.

$$\left.\begin{aligned} f_2(y, 0) &= y \\ f_2(y, 1) &= r_2(y) \end{aligned}\right\} \quad \forall \; y \in N$$

$f_2(b, t) = b$, $b \in B$, $t \in I$.

Since $A \subset M$, then there exist a continuous map $f'(r_1(x)): A \to B$ with $f' = f/A$, $B \subset N$, there exist a continuous map $g'(r_2(y)): B \to A$ with $g' = g/B$, then $f' \circ g' = i_B$ and $g' \circ f' = i_A$ then A, B are of the same homotopy type. To prove the converse, since A is a deformation retract of M, then the inclusion map $i_1 : A \to M$ induces an isomorphism of $\Pi(A, a)$ and $\Pi(M, a)$ for any $a \in A$, $\Pi(A, a)$ is the Poincare group of A at the base point a, also

$i_2: B \to N$ induces an isomorphism of $\Pi(B, b)$ and $\Pi(N, b)$. Since A, B are of the same homotopy type, then there exists continuous maps

f', g' s. t. $f': A \to B$, $g': B \to A$ s .t. $g' \circ f' = i'A : A \to A$ and $f' \circ g' = i'_B: B \to B$, make an extension for f' to f, g' to g, $i'A$ to i_M, i_B' to i_N, then M, N are of the same homotopy type.

REFERENCES

[1] E. EL. KHOLY: Isometric and topological folding of Manifolds, *Ph. D. Thesis* (1981), *University of Southampton, U.K.*

[2] S.A. ROBERTSON: Isometric folding of Riemannian manifolds, *Proc, Roy. Soc. Edinburgh* (77), 1977, 275–284.

[3] W.S. MASSEY: Algebric Topolcgy: An introduction. *Harcourt Brace and Wareld, New York*, (1967).