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"ON BV-REVERSIBLE MATRICES"

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ABSTRACT

In 1932, S. Banach stated that if A is a reversible matrix, then the system of equations

$$Y_n = A_n(x) = \sum_{k=o}^{\infty} a_{nk}x_k$$

has a unique solution given by

$$x = vl + By,$$

where $B = (b_{nk})$ is the unique right inverse of A,

$By = \left(\sum_{k=o}^{\infty} b_{nk}\, y_k\right)_{n=o}^{\infty}$, $y \in c$, $x \in c_A$ and $v = (v_n)_o^{\infty} \in l_\infty$. In 1953 MacPhail showed that v need not belong to l_∞ by giving a simple reversible matrix with v ubounded.

It is the purpose of this paper to extend Banach's work on c-reversible matrices to bv-reversible matrices and construct matrices which are bv-reversible matrices but not c-reversible; the first one with v bounded and the second one with v unbounled.

Notations: s; c; bv; bs; l_∞; c_A ; θ; δ; δ^k; $\bar{X}^*$ will denote the set of all sequences; convergent sequences; sequences of bounded variation that is, sequences such that $\sum_{k=o}^{\infty} |x_{k+1}-x_k| < \infty$ and $\lim_{k\to\infty} x_k$ exists; bounded series, that is, sequences x such that $\sup_{n\geq o} |\sum_{k=o}^{n} x_k| < \infty$; bounded sequences; convergence domain, that is, $c_A = \{x \in s: Ax \in c\}$; $\delta = (1, 1, \ldots)$, $\delta^k = (\delta_n{}^k)^\infty{}_{n=o} = (0,0,\ldots,0,1,0;\ldots)$ $\bar{X}^*$ the continuous dual of X respectively.

1. INTRODUCTION

1.1 **Definition:** (Sequence to sequence)

Let $A = (a_{nk})$, n, k = 0, 1,2... be an infinite matrix of complex numbers. Given a sequence $x = (x_k)_o^\infty$ define

$$y_n = \sum_{k=o}^{\infty} a_{nk} x_k, \; n = 0,1,2, \ldots \tag{1.1}$$

If the series (1.1) converges for all n we call the sequence $(y_n)_o^\infty$ the A – transform of the sequence $(x_k)_o^\infty$. If further $y_n \to a$ as $n \to \infty$ we say that $(x_k)_o^\infty$ is summable by A to a.

1.2. **Definition** (X-reversible)

Let X be an FK – space with a Schauder basis. A matrix $A = (a_{nk})$ is said to be X – reversible if the equation

$$y = Ax \tag{1.2}$$

has a unique solution x, $x \in X_A$, (where $X_A = \{x \in s: Ax = y \in X\}$) for each $y \in X$.

1.3. **Examples** (X = c – reversible matrices)

(i) Normal matrix

(ii) Let $A = (a_{nk})$ be a matrix of the transformation:

$$\left.\begin{aligned} y_{2n} &= \sum_{k=o}^{n} x_{2k} \\ & \qquad\qquad n = 0,1,2\ldots \\ y_{2n+1} &= x_{2n+1} + 2^n \sum_{k=o}^{\infty} x_{2k} \end{aligned}\right\} \tag{1.3}$$

that is,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \ldots \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & \ldots \\ 2 & 0 & 2 & 1 & 2 & 0 & 2 & 0 & & \ldots \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & & \ldots \\ 4 & 0 & 4 & 0 & 4 & 1 & 4 & 0 & & \ldots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\ \ldots & & & \ldots & & & \ldots & & & \end{bmatrix}$$

If $y \in X = c$ we easily see that x is given by:

$$\left.\begin{aligned} x_0 &= y_0 \\ x_{2n} &= y_{2n} - y_{2n-2},\ n \geq 1 \\ x_{2n+1} &= -2^n \lim_{n\to\infty} y_n + y_{2n+1},\ n \geq 0 \end{aligned}\right\} \quad \ldots \qquad (1.4)$$

From equations (1.4) we see that the right inverse of A denoted by $A' = B$ is given by

$$A' = B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & \ldots \\ & \ldots & & & & \ldots & \ldots & & \end{bmatrix}$$

1.4 **Theorem** (Wilansky)

Let $A = (a_{nk})$ be a c-reversible matrix then c_A is a Banach space with norm

$$\| x \| = \sup_{n \geq 0} |y_n|,\ y_n = A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k.$$

The most general continuous linear functional f on c_A is given by

$$f(x) = \alpha \lim y + \sum_{k=0}^{\infty} y_k t_k \qquad (1.5)$$

where $\alpha = F(\delta) - \sum_{k=0}^{\infty} F(\delta^k)$, $t_k = F(\delta^k)$, $t \in l_1$, $f = FoA$, $F \in c^*$.

Proof: See [6] page 229,

1.5 **Theorem** (Wilansky)

Let $A = (a_{nk})$ be a c-reversible matrix then there exists a unique matrix $B = A' = (b_{nk})$ satisfying $\sum_{k=0}^{\infty} |b_{nk}| < \infty$ for each n and a sequence $(v_n)_0^\infty$ such that $y = Ax$ has for each $y \in c$ the unique solution x given by

$$x = vl + By \qquad (1.6)$$

where l = limy, $By = \left(\sum_{k=o}^{\infty} b_{nk} \right)_{n=o}^{\infty}$

Proof. See [6] page 229.

2. **Main Results:**

2.1 **Theorem:** The most general continuous linear functional on bv_o is given by:

$$f(x) = \sum_{k=o}^{\infty} x_k t_k \tag{2.1}$$

where $t_k = f(\delta^k)$, $t \in bs$

2.2 **Theorem:** The most general continuous linear functional on bv is given by:

$$f(x) = l\alpha + \sum_{k=o}^{\infty} (x_k - l) t_k \tag{2.2}$$

where $\alpha = f(\delta)$, $t_k = f(\delta^k)$, $t \in bs$

The proof of 2.1 and 2.2 are elementary see [3].

2.3. **Theorem:** If a matrix $A = (a_{nk})$ is c-reversible, then it is bv-reversible.

Proof: The proof follows easily since A is c-reversible, i.e. A is 1–1 and onto c, but $bv \subset c$ and so A is also 1–1 and onto bv (since bv is a subspace of c) so that for every $y \in bv$ there exists $x \in (bv)_A$ such that $y = Ax$ and vice versa hence $A: (bv)_A \to bv$ is also 1–1 and onto.

2.4. **Theorem:** Let A be a bv-reversible matrix, then there exists a matrix $B = (b_{nk})$, the right inverse of A satisfying for each $n \geq 0$

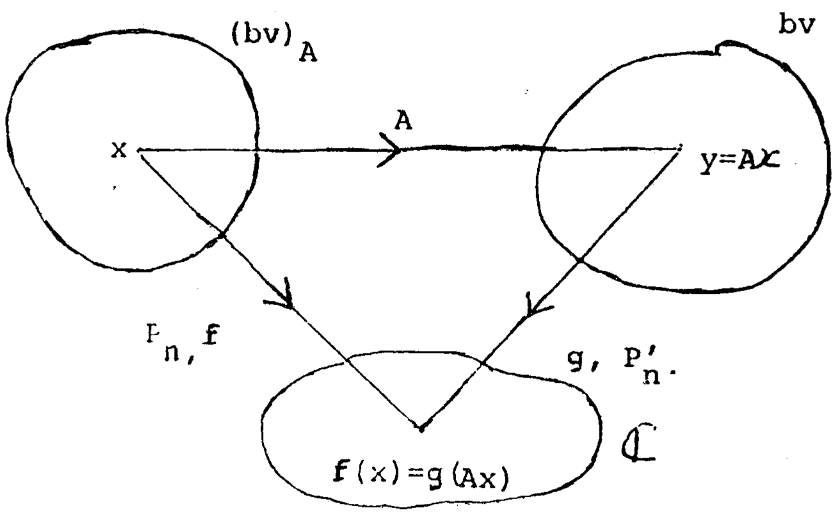
$\sup_{m \geq 0} \left| \sum_{k=o}^{m} b_{nk} \right| = M_n < \infty$ and a sequence $v = (v_n)_o^\infty$ such that

$y = Ax$ has for each $y \in bv$ the unique solution x given by:

$$x = vl + B(y - l\delta) \tag{2.3}$$

where l = limy.

Proof: Let f be a continuous linear functional on $(bv)_A$ and let $A: (bv)_A \to bv$ be a reversible matrix, then $f = goA$ for some $g \in bv^*$



$$f(x) = g(Ax) = g(y) = \alpha \lim y + \sum_{n=0}^{\infty} (y_n - \lim y_n)\, t_n \tag{2.4}$$

where $t_n = g(\delta^n)$, $t \in bs$, $\alpha = g(\delta)$. In particular $P_n \in (bv)^*_A$ since $bv(_A)$ is an FK-space and therefore we have:

$$P_n(x) = x_n = P'_n(\delta) \lim y + \sum_{k=0}^{\infty} (y_k - \lim_{k \to \infty} y_k)\, P'_n(\delta^k)$$

where $P_n = P'_n \circ A$, $P'_n \in bv^*$ (2.5)

$$x_n = v_n \lim y + \sum_{k=0}^{\infty} (y_k - \lim_{k \to \infty} y_k)\, t^n_k$$

Where $v_n = P'_n(\delta)$ and $t^n_k = P'_n(\delta^k) = b_{nk}$, that is,

$$x_n = v_n \lim y + \sum_{k=0}^{\infty} (y_k - \lim_{k \to \infty} y_k)\, b_{nk}, \text{ that is,}$$

$$x = vl + B(y - l\delta).$$

2.5. **Theorem:** If A is c-reversible and hence bv-reversible, then the sequence v′ in $x = v'\, l + B'(y - l\delta)$, where $B' = B$, $v'_n = v_n + B_n(\delta)$ need not be bounded.

Proof: It is enough to give an example. Consider the transformation

$$\left.\begin{aligned} y_{2n} &= \sum_{k=o}^{n} x_{2k} \\ y_{2n+1} &= \frac{1}{1+n} x_{2n+1} - \sum_{k=o}^{\infty} x_{2k} \end{aligned}\right\} n=0,1,2,\ldots \qquad (2.6)$$

i.e.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & -1 & \frac{1}{3} & -1 & 0 & -1 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ -1 & 0 & -1 & 0 & -1 & \frac{1}{3} & -1 & 0 & \ldots \\ & \ldots & & & \ldots & & \ldots & & \end{bmatrix}$$

Solving for x we obtain:

$$\left.\begin{aligned} x_o &= y_o \\ x_1 &= y_1 + \lim_{n\to\infty} y_{2n} \\ x_2 &= y_2 - y_o \\ x_{2n} &= y_{2n} - y_{2n-2},\ y_{-2} = 0 \\ x_{2n+1} &= (n+1)\,(y_{2n+2} + \lim_{n\to\infty} y_{2n}) \end{aligned}\right\} \qquad (2.7)$$

From this solution for x we see that B is given by:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & \ldots \\ & \ldots & & & & \ldots & & \ldots & \end{bmatrix}$$

From (2.7) it is clear that $v_n = n+1$, which is unbounded as $n \to \infty$.

$$\begin{aligned} \text{Now}\quad v'_n &= v_n + B_n(y) \\ &= (n+1) + B_n(y) \\ v' &= (n+1) + (1,1,0,2,0,3,0,4,0,\ldots) \\ &= (2,3,3,6,5,9,7,12,9,11,18,\ldots) \end{aligned}$$

i.e.

$$\left.\begin{aligned} v'_o &= 2 \\ v'_{2n+1} &= 3\,(n+1),\ n = 0,1,2,.. \\ v'_{2n} &= 2\,n + 1,\ n = 1,2,3,\ldots \end{aligned}\right\} \qquad (2.8)$$

which is unbounded.

We are now in a position to give an example to show that a bv-reversible matrix need not be c-reversible.

2.6. **Theorem:** There exists a matrix which is bv-reversible but not c-reversible.

Proof: It is enough to give an example. We know that if A is c-reversible, i.e $A: c_A \to c$ is 1–1 and onto c, then the unique solution of the equation $y = Ax$ is given by $x = vl + By$, where $\sum_{k=0}^{\infty} |b_{nk}| < \infty$ for each $n \geq 0$ see theorem 1.5. So let us consider the matrix $A' = B = (b_{nk})$ given by:

$$
\begin{bmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & \dots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & \dots \\
\dots & & & \dots & & \dots & &
\end{bmatrix}
$$

This matrix is chosen so that $\sum_{k=0}^{\infty} |b_{nk}|$ will not exist for at least one n. In particular $\sum_{k=0}^{\infty} |b_{nk}|$ does not exist for $n = 0$ and therefore the matrix A if it exists such that $AB = I$ cannot be c-reversible (see theorem 1.5).

To determine the matrix A we row-reduce the matrix below in echelon form:

$$
\left[\begin{array}{cccccccc|cccccccc}
1 & -1 & 1 & -1 & 1 & -1 & 1 & \dots & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\
\dots & & \dots & & \dots & & & & \dots & & \dots & & \dots & &
\end{array}\right]
$$

$$
\left[\begin{array}{cccccccc|ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & \dots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\
\dots & & \dots & & \dots & & & & \dots & & \dots & & \dots & & & &
\end{array}\right]
$$

Therefore we have:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ & \dots & & & \dots & & & \dots & \end{bmatrix}$$

By straightforward calculations one quickly checks that $AB = I$. We still need to prove that A is bv-reversible i.e. the equation $y = Ax$ has a unique solution $x \in (bv)_A$ for each $y \in bv$ given by:

$$\left.\begin{aligned} x_0 &= \sum_{n=o}^{\infty} (y_{2n} - y_{2n+1}) \\ x_{2n} &= y_{2n+1}.\ n = 1,2,\dots \\ x_{2n+1} &= y_{2n+1},\ n = 0, 1,2,\dots \end{aligned}\right\} \tag{2.9}$$

where the transformation $y = Ax$ is given by:

$$\left.\begin{aligned} y_0 &= x_0 + x_1 - \sum_{k=1}^{\infty} x_{2k} \\ y_{2n} &= x_{2n} + x_{2n+1},\ n = 1,2,\dots \\ y_{2n+1} &= x_{2n+1},\ n = 0,1,2,\dots \end{aligned}\right\} \tag{2.10}$$

We need to show that x_0 exists, but this is obvious since

$$\sum_{n=o}^{\infty} (y_{2n} - y_{2n+1})$$

is absolutely convergent hence convergent

$$\left(\text{since } \sum_{n=o}^{\infty} |y_n - y_{n+1}| < \infty \text{ for } y \in bv\right).$$

A is clearly 1–1 since Ker (A) $= \{\theta\}$

Now x is uniquely determined for $y \in bv$ and so A is onto. But A is not onto c since if $y_n = \dfrac{(-1)^n}{1 + n}$, then there is no $x \in c_A$ such that $y = Ax$ since $\sum_{n=o}^{\infty} (y_{2n} - y_{2n+1})$ diverges. Therefore A is bv-reversible but not c-reversible.

We now show that v in $v = vl + B(y - l\delta)$ is bounded for this A.

$$x_0 = v_0 \lim y + \sum_{k=o}^{\infty} b_{ok} (y_k - \lim y) \equiv \sum_{n=o}^{\infty} (y_{2n} - y_{2n+1})$$

$$= v_0 \lim y + \sum_{k=o}^{\infty} (-1)^k (y_k - \lim y) \equiv \sum_{n=o}^{\infty} (y_{2n} - y_{2n+1})$$

Let $(-1)^k = s_k - s_{k-1}$, then $\sum_{k=o}^{n} (-1)^k = s_n$, $s_{2n} = 1$,

$S_{2n+1} = 0$. We can now write $\sum_{k=o}^{\infty} (-1)^k (y_k - \lim y)$ as:

$\lim_{N \to \infty} \sum_{k=o}^{N} (s_k - s_{k-1}) (y_k - \lim y)$. Summation by parts gives,

$$\sum_{k=o}^{\infty} (-1)^k (y_k - \lim y) = \lim_{N \to \infty} \left[\sum_{k=o}^{N-1} S_k (y_k - y_{k+1}) + S_N (y_N - \lim y) \right]$$

i.e. $\sum_{k=o}^{\infty} (-1)^k (y_k - \lim y) = \sum_{k=o}^{\infty} S_{2k} (y_{2k} - y_{2k+1})$ as $N \to \infty$

$$\therefore x_0 = v_0 \lim y + \sum_{k=o}^{\infty} s_{2k} (y_{2k} - y_{2k+1}) = \sum_{k=o}^{\infty +} (y_{2k} - y_{2k+1})$$

$\Rightarrow v_0 \lim y = 0 \Rightarrow v_0 = 0$

$$x_1 = v_1 \lim y + \sum_{k=o}^{\infty} b_{1k} (y_k - \lim y) \equiv y_1$$

i.e. $x_1 = v_1 \lim y + \sum_{k=o}^{\infty} (y_k - \lim y) \equiv y_1$ since $b_{1k} = 0$

for $k \neq 1$, $b_{11} = 1$ we see that $v_1 = 1$. Similarly $v_2 = 0$, $v_3 = 1$, ... Therefore $v = (v_n)_o^{\infty} = (0, 1, 0, 1, 0, \ldots)$ which is bounded.

2.7. **Theorem:** Let A be a bv-reversible matrix which is not c-reversible, then the sequence v in $v = v\, l + B (y - l\, \delta)$ need not be bounded.

Proof: It is enough to give an example. Consider the transformation given by:

$$\left.\begin{aligned} y_0 &= x_0 + \sum_{k=1}^{\infty} \frac{1}{2k} x_{2k-1} \\ y_{2k} &= \sum_{\nu=k}^{\infty} (-1)^{\nu-k} \frac{1}{1+2\nu} x_{2\nu},\ k>0 \\ y_{2k+1} &= \frac{1}{2+2k} x_{2k+1} + \sum_{\nu=k+1}^{\infty} (-1)^{\nu-k-1} \frac{1}{1+2\nu} x_{2\nu},\ k\geq 0 \end{aligned}\right\} \qquad (2.11)$$

That is,

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{6} & 0 & \frac{1}{8} & 0 & \frac{1}{10} & 1 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{-1}{5} & 0 & \frac{1}{7} & 0 & \frac{-1}{9} & 0 & \frac{1}{11} & \cdots \\ 0 & 0 & \begin{matrix}1\\3\\3\end{matrix} & 0 & \frac{-1}{5} & 0 & \frac{1}{7} & 0 & \frac{-1}{9} & 0 & \frac{1}{11} & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{5} & 0 & \frac{-1}{7} & 0 & \frac{1}{9} & 0 & \frac{-1}{11} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{-1}{7} & 0 & \frac{1}{9} & 0 & \frac{-1}{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{7} & 0 & \frac{-1}{9} & 0 & \frac{1}{11} & \cdots \\ & \cdots & & \cdots & & & & & \cdots & & & \end{bmatrix}$$

A is bv-reversible but not c-reversible. To see this, the unique solution of the equation $y = Ax$ is given by

$$\left.\begin{aligned} x_0 &= y_0 - \sum_{k=1}^{\infty} (y_{2k-1} - y_{2k}) \\ x_{2k} &= (2k+1)\,(y_{2k} + y_{2k+1},\ k\geq 1 \\ x_{2k-1} &= 2k\,(y_{2k-1} - y_{2k}),\ k\geq 1 \end{aligned}\right\} \qquad (2.12)$$

provided x_0 exists, that is, $\sum_{k=1}^{\infty} (y_{2k-1} - y_{2k})$

is convergent, but $y \in bv \Rightarrow \sum_{k=0}^{\infty} |y_{k+1} - y_k| = \sum_{k=0}^{\infty} |y_k - y_{k+1}| < \infty$

But also $\sum_{k=0}^{\infty} |y_k - y_{k+1}| = \sum_{k=0}^{\infty} |y_{2k} - y_{2k+1}| + \sum_{k=1}^{\infty} |y_{2k} - y_{2k-1}| < \infty$

$$\Rightarrow \sum_{k=1}^{\infty} |y_{2k} - y_{2k-1}| < \infty \Rightarrow \sum_{k=1}^{\infty} (y_{2k} - y_{2k-1}) \text{ is convergent.}$$

The right inverse of A denoted by B or A′ is given by:

$$B = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \dots \\ 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & -4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 5 & 0 & 5 & 0 & \dots \\ & \dots & & & \dots & & \dots & & \end{bmatrix}$$

Clearly AB = I. A c-reversible $\Rightarrow \sum_{k=o}^{\infty} |b_{nk}| < \infty$ for all n,

$(b_{nk})_{k=o}^{\infty} \in bs$, but $\sum_{k=o}^{\infty} |b_{nk}| = \infty$ for n = 0 so A is not c-reversible.

Also $y_n = (-1)^n \frac{1}{1+n}$, $y \in c$ but $\sum_{n=o}^{\infty} (y_{2n-1} - y_{2n})$ diverges. Since A is bv-reversible, $x_n = V_n' \, l + B_n (y - l\delta)$. In particular we have

$$x_o = V_o \, l + \sum_{k=o}^{\infty} (-1)^k (y_k - l)$$

$$= V_o \, l + \lim_{k \to \infty} \sum_{k=o}^{2k} (-1)^k (y_k - l), \text{ i.e.}$$

$$\sum_{k=o}^{\infty} (-1)^k (y_k - l) = \lim_{k\to\infty} (y_o - l) - (y_1 - l) + (y_{2k} - y_{2k-1})$$

$$= (y_o - l) + \sum_{k=1}^{\infty} (y_{2k} - y_{2k-1})$$

$$x_o = v_o \, l + (y_o - l) + \sum_{k=1}^{\infty} (y_{2k} - y_{2k-1}) \qquad (2.13)$$

Comparing (2.13) with

$x_o = y_o - \sum_{k=1}^{\infty} (y_{2k-1} - y_{2k})$ we get

$v_o \, l = l \Rightarrow v_o = 1$ if $l \neq 0$. Next we have:

$$x_1 = v_1 l + \sum_{k=o}^{\infty} b_{1k} (y_k - l)$$
$$= v_1 l + 2 (y_1 - l) - 2 (y_2 - l)$$
$$= 2 (y_1 - y_2) \Rightarrow v_1 = 0$$

v_2, v_3, v_4 ... are obtained in the same way as v_1 giving

$v = (1,0,6,0,10,0,14,0,18, ...)$, i.e.

$$\left.\begin{array}{l} v_o = 1 \\ v_{2n} = 4n+2, n \geq 1, \\ v_{2n+1} = 0, n \geq 0 \end{array}\right\} \qquad (2.14)$$

wich is unbounded as $n \to \infty$

2.8. Remarks:

(i) If A is c-reversible then it is bv-reversible (see theorem 2.3); its right inverse need not be its left inverse.

Proof: It is enough to give an example. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 0 & 1 & 1 & \dots \\ 1 & 0 & 0 & 1 & \dots \\ & \dots & \dots & & \end{bmatrix}$$

then A is c-reversible (See [6])

hence bv-reversible. Its right inverse A′ is given by

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & & \dots \\ 1 & -1 & 0 & 0 & & \dots \\ 0 & 1 & -1 & 0 & & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ & \dots & & \dots & & \dots \end{bmatrix}$$

See [5], [6]

AA′ = I. 'A the left inverse of A does not exist since if it existed we would have:

$$\left.\begin{array}{l} \sum_{k=o}^{\infty} b_{nk} = 1 \text{ for } n=0 \\ b_{no} = 0 \\ b_{no} + b_{n1} = 0 \\ b_{no} + b_{n1} + b_{n2} = 0 \end{array}\right\} \Rightarrow b_{oo} = b_{o1} = b_{o1} = \dots$$

contradicting the fact that $\sum^{\infty} b_{nk} = 1$ for $n=0$ so that $'AA \neq I$ if $'A$ exists and hence $'A \neq A'$

(ii) If a matrix A has a two-sided inverse A^{-1}, then it need not be c-reversible

Proof. It is enough to give an example. Let A be the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 & 0 & \dots \\ 0 & -1 & 2 & 0 & 0 & \dots \\ 0 & 0 & -1 & 2 & 0 & \dots \\ & \dots & & \dots & \dots & \end{bmatrix}$$

then A^{-1} the inverse of A is given by.

$$A^{-1} = \begin{bmatrix} -1 & -2 & -4 & -8 & -16 & \dots \\ 0 & 1 & 2 & 4 & 8 & \dots \\ 0 & 0 & -1 & -2 & -4 & \dots \\ & \dots & & \dots & \dots & \end{bmatrix}$$

Now A; $c_A \to c$ is not 1–1 since it transforms both v and $\left(\frac{1}{2^n}\right)_0^\infty$ to θ so it is not 1–1 and onto bv either. Hence A is not c-reversible, it is not bv-reversible. See also [5].

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