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ON THE RESTRICTED SHEAF

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SUMMARY

Let X be a connected complex analytic manifold of dimension n with fundamental group $H_x \neq \{1\}$, H be the Sheaf of the fundamental groups over X [1], $\Gamma(X, H)$ be the group of the global sections of H over X and $[H, H] \subset H$ be the Commutator Subsheaf. It is shown that the Commutator Subsheaf [H, H] (or equivalently the Commutator subgroup $\Gamma(X, [H, H])$ of $\Gamma(X, H)$) determines the Restricted Sheaf A of germs of holomorphic functions over X [5] and the subsheaves of H defined by the normal subgroups of $\Gamma(X, H)$ such that they contain [H, H] (or the normal subgroups of $\Gamma(X, H)$ such that they contain $\Gamma(X, [H, H])$) determine the Restricted Ideal Sheaves of the sheaf A [6]. In addition, the subsheaves of H (or the normal subgroups of Γ (X, H)), which determine the Restricted Ideal Sheaves, satisfy the descending (minimal) chain condition.

Finally, the Commutator Subgroup Γ (X, [H, H]) completely determines the vector space A (X) of holomorphic functions on X.

1. INTRODUCTION

Let X be a connected complex manifold of dimension n with fundamental group $H_X \neq \{1\}$ (or a connected, locally arcwise connected Hausdorff Space), H_X be the fundamental group of X with respect to the point x, for any $x \in X$, $c \in X$ be an arbitrarily fixed point and X = (X,c) be a pointed space. Let $H = \bigvee_{x \in X} H_X$. H is a set over X and the mapping φ: H → X such that $\varphi(\sigma_X) = \varphi[\alpha]_X) = x$ for any $\sigma_X = [\alpha]_X \in H_X \subset H$ is onto.

We introduce on H a natural topology as follows.

Let $x \in X$ be any point and W = W (x) be an open neighborhood of x. Let us define a mapping s: W → H such that $s(x) = [(\gamma^{-1}\alpha)\gamma]_X$ for any $x \in W$, where $\sigma_C = [\alpha]_C \in H_C$ is an element and $[\gamma]$ is the homotopy class fedines an isomorphism between H_X and H_C. The homotopy class $[\Upsilon]$ is choosen arbitrarily fixed. Hence, $s = s(\sigma_c)$. Moreover, φ $os = 1_W$. Let us denote the totality of the mappings s by $\Gamma(W,H)$. If B is a basis of arcwise connected open neighborhoods for X, then $B^* = \{s(W): W \in B, s \in \Gamma(W,H)\}$ is a basis for H. In this topology, the mappings φ and s are continuous. Furthermore, φ is a locally topological mapping. Thus, (H, φ) is a sheaf over X. (H, φ), or only H, is called The Sheaf of the Fundamental Groups over X [1].

Let $W \subset X$ be an open set. A continuous mapping s from W to H can be defined in similar manner. The mapping s is called a section of H over W. The set $\Gamma(W,H)$ is a group. Thus, H is a sheaf of groups over X [3]. In addition, the group H_x is called a stalk of H over, x, for any $x \in X$.

The Sheaf H satisfies the following properties.

1. $W \subset X$ be an open set. Then, a section over W can be extended to a global section over X. Furthermore,

$$\Gamma(W,H) = \Gamma(s(W),H),\ s \in \Gamma(X,H).$$

2. Any two stalks of H are isomorphic with each other.

3. Let $W_1, W_2 \subset X$ be any two open sets, $s_1 \in \Gamma(W_1, H)$ and $s_2 \in \Gamma(W_2, H)$. If $s_1(x_o) = s_2(x_o)$, for any point $x_o \in W_1 \cap W_2$, then $s_1 = s_2$ over the whole $W_1 \cap W_2$.

4. Let $W \subset X$ be an open set and $s_1, s_2 \in \Gamma(W,H)$. If $s_1(x_o) = s_2(x_o)$ for any point $x_o \in W$, then $s_1 = s_2$ over the whole W.

5. Let $x \in X$ be any point and $W = W(x)$ be a open set. Then, $\varphi^{-1}(W) = \bigvee_{i \in I} s_i(W)$ for every $s_i \in \Gamma(W,H)$ and $\varphi \mid s_i(W): s_i(W) \to W$ is a topological mapping for each $i \in I$. Thus, H is a covering space of X [4], such that to each point $\sigma_x \in H_x$ there corresponds a unique section $s \in \Gamma(W,H)$ such that $s(x) = \sigma_x$. Furthermore, H_x is isomorphic to $\Gamma(W,H)$. In particular, $H_x \cong \Gamma(X,H)$.

6. A topological stalk preserving mapping of H onto itself is called a sheaf isomorphism or a cover transformation, and the set of all cover transformation of H is denoted by T. Clearly, T is a group. It can be proved that T is isomorphic to the group $\Gamma(X,H)$. Hence, $H_x \cong \Gamma(X,H) \cong T$. Thus, T is transitive and X is a regular covering space of X [4].

2. THE GROUP $\Gamma(X,H)$ AND SUBSHEAVES OF H.

We begin by giving the following definition [3].

Definition 2.1. Let H be the sheaf of fundamental groups over X and $H' \subset H$ be an open set. Then H' is called a subsheaf of groups, if:

i) $\varphi(H') = X$

ii) For each point $x \in X$ the stalk H'_x is a subgroup of H_x.

We now give the following definition.

Definition 2.2. Let H be the sheaf of the fundamental groups over X and $N \subset H$ be a subsheaf of groups. Then N is called a normal subsheaf, if the stalk $N_x \subset H_x$ is a normal subgroup for each $x \in X$.

Let $H' \subset H$ be a subsheaf of groups and $W \subset X$ be an open set. Then, the set $\Gamma(W,H') \subset \Gamma(W,H)$ is a subgroup. Moreover, if $N \subset H$ is a normal subsheaf, then $\Gamma(W,N) \subset \Gamma(W,H)$ is a normal subgroup. In particular, if we take $W = X$, then $\Gamma(X,N) \subset \Gamma(X,H)$ is a normal subgroup. Consequently, each subsheaf of groups gives a subgroup of $\Gamma(X,H)$ and each normal subsheaf gives a normal subgroup of $\Gamma(X, H)$.

Conversely, let us suppose that, $\Gamma(X,H)$ is the group of global sections of H over X and $D \subset \Gamma(X,H)$ be a subgroup.

Then, the set $\{s_i(x): s_i \in D\}$ is a subgroup of H_x over x for each $x \in X$. Let us denote $\{s_i(x): s_i \in D\}$ by H'_x. Then $H' = \bigvee_{x \in X} H'_x$ is a over X with the natural projection $\varphi' = \varphi \mid H'$ and $D = \Gamma(X,H')$. Moreover, if $D \subset \Gamma(X,H)$ is a normal subgroup, then each stalk of H' is a normal subgroup of H_x. One can show that (H', φ') is a subsheaf of groups and (H', φ') is a normal subsfheaf of H, if $D \subset \Gamma(X,H)$ is a normal subgroup.

Then we may state,

Theorem 2.1. Let H be the sheaf of the fundamental groups over X and $\Gamma(X,H)$ be the group of global sections of H over X. Then, the subgroups of $\Gamma(X,H)$ define all the subsheaves of groups of H. In particular, a normal subgroup of $\Gamma(X,H)$ defines a normal subsheaf of H.

It is easily seen that, subsheaves of groups of H (or normal subsheaves of H) have all the properties of H stated in introdiuction Thus, they are also regular covering spaces of X and $H'_x \cong \Gamma(X,H) = T'$ for each subsheaf of groups $H' \subset H$.

Let $D', D'' \subset \Gamma(X,H)$ be any two subgroups and $D' \subset D''$,. Then, $H' \subset H''$ for the corresponding sheaves H' and H'', respectively Furthermore, the ordering $D' \subset D''$ ... of normal subgroups of $\Gamma(X,H)$ given by inclusion is isomorphic with the ordering $N' \subset N''$ ... of normal subsheaves of H by inclusion. Moreover, each chain of normal subsheaves has an upper bound. By Zorn's lemma there is a maximal normal subsheaf which is H corresponds to $\Gamma(X,H)$

Definition 2.3. Let $D \subset \Gamma(X,H)$ be the commutator subgroup. The normal subsheaf defined by D is called the Commutator Subsheaf of H and it is denoted by $[H,H]$.

Clearly, $D = \Gamma(X, [H,H])$. In addition, the ordering $D = \Gamma(X, [H,H]) \subset D' \subset D''$ ... of normal subgroup of $\Gamma(X,H)$ given by inclusion is isomorphic with the ordering $[H,H] \subset N' \subset N''$ ... of normal subsheaves of H by inclusion.

We now give the following propositions.

Proposition 2.1. Let H be the sheaf of the fundamental groups and $H', H'' \subset H$ be any two subsheaves of groups. Then, $H' \subset H''$ is a subsheaf of groups if and only if $\Gamma(X,H') \subset \Gamma(X,H'')$.

As a definition, if $H' \subset H''$ is a subsheaf of groups, then it is said that H'' is stronger than H'.

Proposition 2.2. Let H be th sheaf of the fundamental groups over X and $N \subset H$ be a normal subsheaf. Then $N \supset [H,H]$ if and onlyif $\Gamma(X, [H,H]) \subset \Gamma(X,N)$.

3. SUBSHEAVES AND QUOTIENT SHEAVES OF H.

In this section, we will study the relationship among the subgroups of $\Gamma(X,H)$, the subsheaves of H and the quotient sheaves of H.

Let H be the sheaf of the fundamental groups and $H' \subset H$ be a subsheaf of groups. Let us associate the set $M_W = \Gamma(W,H)/\Gamma(W,H')$ with the open set W, for each $W \subset X$ open. Then, the system $\{X, M_W = \Gamma(W,H)/\Gamma(W,H'), \gamma_{W,V}\}$ is a pre-sheaf [2]. The sheaf defined by the pre-sheaf $\{X, M_W, \gamma_{W,V}\}$ is called Quotient Sheaf and it is denoted by Q_H'.

We now give the following theorem.

Theorem 3.1. Let H be sheaf of the fundamental groups over X, $H' \subset H$ be a subsheaf of groups and Q_H' be the corresponding sheaf. Then the sheaf Q_H' is a sheaf of groups over X if and only if H' is a normal subsheaf of H (it is not necessary to contain the Commutator Subsheaf [H,H]).

Proof. Let $H' \subset H$ be subsheaf of groups and Q_H' be the corresponding quotient sheaf. Let us suppose that Q_H' is a sheaf of groups over X. $Q_H' = \bigvee_{x \in X} (Q_H')_x$ and

$(Q_H')_x = \{(W, [s])_x$: $W \subset X$ is an open set, $[s] \in \Gamma(W,H)/\Gamma(W,H')\}$ Furthermore, each stalk $(Q_H')_x$, for each $x \in X$, and the set $\Gamma(X, Q_H')$, are groups. The group operation defined in each stalk is as follows:

$(W, [s_1])_x . (W, [s_2])_x = (W, [s_1.s_2])_x$, for every $(W, [s_1])_x$, $(W, [s_2])_x \in (Q_H')_x$

The operation does not depend on the representatives of equivalent classes, because it is well-defined. So, s. $\Gamma(X,H') = \Gamma(X,H)$. s for each $s \in \Gamma(X,H)$. Thus, $\Gamma(X,H') \subset \Gamma(X,H)$ is a normal subgroup, and $H' \subset H$ is a normal subsheaf.

Conversely, let us suppose that $H' \subset H$ is a normal subsheaf over X. So, $\Gamma(X,H') \subset \Gamma(X,H)$ is a normal subgroup. Thus, $\Gamma(X,H)/\Gamma(X,H')$ is a group. So, the operation defined in each stalk $(Q_H')_x$ in the form of $(W, [s_1])_x (W, [s_2])_x = (W, [s_1.s_2])_x$ is well-defined. It is easily seen that each stalk $(Q_H')_x$ is a group with this operation for every $x \in X$. Moreover $\Gamma(X,Q_H')$ is a group. So, Q_H' is a sheaf of groups.

One can prove that, the group $\Gamma(X, Q_H')$ is isomorphic to the group $(Q_H')_x$.

Theorem 3.2. Let H be the sheaf of the fundamental groups, $N \subset H$ be a normal subsheaf and Q_N be the corresponding quotient sheaf. Then the group $\Gamma(X, Q_N)$ is isomorphic to the quotient group $\Gamma(X,H)/\Gamma(X,N)$.

Proof. To prove this theorem, let us define the mapping γ: $\Gamma(X,H)/\Gamma(X,N) \to \Gamma(X,Q_N)$ in the form of $\gamma([s]) = \gamma[s]$, where γ representes inductive limit [2]. If, $\gamma([s]) = 1$, then $\gamma[s] = 1$ and so, $\gamma[s](x) = (X, [e])_x$, for any $x \in X$. That is, $(W, [s])_x = (W, [e])_x$. Thus, $[s] = [e]$ Hence, γ is one-to-one. Clearly γ is onto. Now, if $[s_1]$, $[s_2] \in \Gamma(X,H)/\Gamma(X,N)$ are any two elements, then

$\gamma([s_1].[s_2]) = \gamma([s_1.s_2]) = \gamma[s_1.s_2] = \gamma[s_1].\gamma[s_2]$. Thus, γ is a homomorphism.

Therefore, γ: $\Gamma(X,H)/\Gamma(X,N) \to \Gamma(X,Q_N)$ is an isomorphism.

Then, we may state the following theorem which is a criterion on the sheaf of Abelian groups [7].

Theorem 3.3. Let H be the sheaf of fundamental groups, and $N \subset H$ be a normal subsheaf. Then, N determines the sheaf Q_N of groups over X. The sheaf Q_N is a sheaf of Abelian groups over X if and only if $N \supset [H,H]$.

One can show that, the sheaf Q_H' and Q_N are regular covering spaces of X [7]. Then, we can state by considering proposition 2.2 and Theorem 3.3.

Theorem 3.4. Let H be the sheaf of the fundamental groups and $\Gamma(X,H)$ be the group of global sections of H over X. Then, each normal subgroup $\Gamma(X,N)$ of $\Gamma(X,H)$ determines a sheaf of groups over X which is a regular covering space of X. The sheaf determined by $\Gamma(X,N)$ is a sheaf of Abelian groups if and only if $\Gamma(X,N) \supset \Gamma(X, [H,H])$.

By considering Theorem 3.4, we can state the following criterion [7].

Criterion. Let H be the sheaf of the fundamental groups, $\Gamma(X,H)$ be the group of global sections of H over X and $\Gamma(X,N) \subset \Gamma(X,H)$ be a normal subgroup. Then, $\Gamma(X,H)/\Gamma(X,N)$ is commutative (or a sheaf of groups over X determined by $\Gamma(X,N)$ is a sheaf of Abelian groups) if and only if $\Gamma(X,N) \subset \Gamma(X, [H,H])$.

Let $N', N'' \subset H$ be normal subsheaves and $N', N'' \supset [H,H]$. If $N' \subset N''$, then $\Gamma(X,N') \subset \Gamma(X,N'')$. Now, if the mapping $\Phi: \Gamma(X,H)/\Gamma(X,N'') \to \Gamma(X,H)/\Gamma(X,N')$ is defined by $\Phi(s.N'') = s.N'$, for each $s \in \Gamma(X,H)$, then Φ is a monomorphism. So, we identify $\Gamma(X,H)/\Gamma(X, N'')$ with $\Phi(\Gamma(X,H)/\Gamma(X,N''))$. Thus, we can write

$\Gamma(X,H)/\Gamma(X,N'') \subset \Gamma(X,H)/\Gamma(X,N')$. Now, if $N', N'' \subset H$ normal subsheaves such that $N',N'' \supset [H,H]$ and Q_N', Q_N'' are corresponding sheaves, respectively, then $Q_N' \supset Q_N''$.

Therefore, an ordering $N' \subset N''$ ... of normal subsheaves of H such that they contain the Commutator Subsheaf [H,H] given by inclusion is isomorphic with the ordering $Q_N' \subset Q_N''$ ... of the sheaves of Abelian groups by inclusion. Moreover, each chain of the sheaves of Abelian groups has an upper bound. By Zorn's lemma there is a maximal sheaf $Q_{[H,H]}$ of Abelian groups corresponds to the Comutator subsheaf [H,H].

Let $\Gamma(X,H)$ be the group of global section of H over X and $\Gamma(X,N) \subset \Gamma(X,H)$ be a normal subgroup. $\Gamma(X,N)$ is called proper normal subgroup, if $\Gamma(X,N)$ is different from $\{1\}$ and $\Gamma(X,H)$. Then we have [6], Theorem 3.5. Let $\Gamma(X,N) \subset \Gamma(X,H)$ be a proper normal subgroup such that $\Gamma(X,N) \neq \Gamma(X, [H,H])$ and $\Gamma(X,H)/\Gamma(X,N)$ is commutative. Then, there exists a normal subgroup $\Gamma(X,N')$ with same qualification such that $\Gamma(X,N) \supset \Gamma(X,N')$. Namely, these $\Gamma(X,N)$'s satisfy the descending (minimal) chain condition.

If we define a proper normal subsheaf as being different from H and 1 (1 is identy sheaf) we may state,

Theorem 3.6. Let $N \subset H$ be a proper normal subsheaf such that $N \supset [H,H]$ and $N \neq [H,H]$. Then there exists a normal subsheaf N' with same qualification such that $N \supset N'$. Namely, these N's satisfy the descending (minimal) chain condition.

We summarize this section by stating the following equivalent theorems.

Theorem 3.7. Let X be a connected complex manifold of dimension n with fundamental group $H_X \neq \{1\}$ (Or a connected locally arwise connected Hausdorff space), H be the sheaf of the fundamental groups over X and $N \subset H$ be a normal subsheaf such that $N \supset [H,H]$. Then N determines a sheaf of Abelian groups over X which is a regular covering space of X. The Commutator subsheaf $[H,H]$, which is the smallest normal subsheaf of that type, determines the maximal sheaf $Q_{[H,H]}$.

Or equivalently,

Theorem 3.7.* Let X be a connected complex manifold of dimension n with fundamental group $H_X \neq \{1\}$ (or a connected locally arcwise connected Hausdorff space), H be the sheaf of the fundamental groups over X, $\Gamma(X,H)$ be the group of global sections of H over X and $D \subset \Gamma(X,H)$ be a normal subgroup such that $D \supset \Gamma(X, [H,H])$. Then, D determines a sheaf of Abelian groups over X which is a regular covering space of X. The commutator subgroup $\Gamma(X, [H,H])$, which is the smallest normal subgroup of that type, determines the maximal sheaf $Q_{[H,H]}$

4. QUOTIENT SHEAF $Q_{[H,H]}$ AND RESTRICTED SHEAF A.

Let X be a connected complex manifold of dimension n with funmental group $H_X \neq \{1\}$, $c \in X$ be an arbitrary fixed point, H_c be the fundamental group of X with respect to the point c and $H'_c \subset H$ be a subgroup. For each $x \in X$, H_X isomorphic to H_c and H'_c is isomorphic to a subgroup H'_x of H_X. Then, H_X/H'_x is isomorphic to H_c/H'_c. Let $\overline{H}_H' = \bigvee_{x \in X} H_X/H'_x$. $\overline{H}_H'$ is a set over H with the naturalprojection $\bar{\varphi}: \overline{H}_H' \to X$

defined by $\bar{\varphi}(\bar{\sigma}_x) = \bar{\varphi}([\bar{\alpha}]_x) = x$, for any $\bar{\sigma}_x = [\bar{\alpha}]_x \varepsilon (\bar{H}_{H'},)_x \subset \bar{H}_{H'}$. We introduce on $\bar{H}_{H'}$, a natural topology in similar manner to the topology of H. In this topology, $\bar{\varphi}$ is a locally topological mapping. Then $(\bar{H}_{H'}, \bar{\varphi})$ is a sheaf over X.

Now, let $W \subset X$ be an open set. A section $\bar{s}$ from W to $\bar{H}_{H'}$, is defined by $\bar{s}(x) = \overline{s(x)}$ for each $x \in W$, where $s \in \Gamma(W,H)$. $\bar{s}$ is continuous. The totality of sections over W is denoted by $\bar{\Gamma}(W,\bar{H}_{H'})$. Let $x \in X$ be any point and $W = W(x)$ be an arcwise connected open neighborhood of x. Then $\bar{\varphi}^{-1}(W) = \bigvee_{i \in I} \bar{s}_i(W)$, for every $\bar{s}_i \in \bar{\Gamma}(W,\bar{H}_H)$ and $\bar{\varphi} \mid \bar{s}_i(W): \bar{s}_i(W) \to W$ is a topological mapping. Thus, each open set $W = W(x)$ is evenly covered by $\bar{\varphi}$. Then, $(\bar{H}_{H'}, \bar{\varphi})$ is a covering space of X. It can be shown that $\bar{H}_{H'}$ is a regular covering space.

Let $N_c \subset H_c$ be a normal subgroup. Then, the corresponding sheaf $\bar{H}_N$ is a sheaf of groups. The sheaf $\bar{H}_N$ satisfies the similar properties which stated in introduction for the sheaf H. Let $[H_c, H_c] \subset H_c$ be the commutator subgroup and $N_c \subset H_c$ be a normal subgroup. Then, H_c/N_c is commutative if and only if $N_c \supset [H_c,H_c]$. Thus, $\bar{H}_N$ is a sheaf of Abelian groups over X if and only if $N_c \supset [H_c,H_c]$. Moreover, if N'_c, $N''_c \subset H_c$ are any normal subgroups such that $N_c' \subset N''_c$, and $\bar{H}_{N'}$, $\bar{H}_{N''}$ are corresponding sheaves, then we can write $\bar{H}_{N'} \supset \bar{H}_{N''}$ by using the identification method. Thus, an ordering $N_c' \supset N''_c \ldots$ of normal subgroups of H_c by inclusion is isomorphic with the ordering $\bar{H}_{N'} \supset \bar{H}_{N''} \ldots$ of the corresponding sheaves of groups by inclusion. In particular, an ordering $[H_c,H_c] \subset N'_c \subset N''_c \ldots$ of normal subgroups of H_c by inclusion is isomorphic with the ordering $\bar{H}_{[H_c,H_c]} \supset \bar{H}_{N'} \supset \bar{H}_{N''} \ldots$ of the corresponding sheaves of Abelian groups. Moreover, each chain of corresponding sheaves of Abelian groups has an upper bound. By Zorn's lemma, there is a maximal sheaf of Abelian groups that is $\bar{H}_{[H_c, H_c]}$ corresponds to $[H_c,H_c]$. The sheaf $\bar{H}_{[H_c, H_c]}$ is called the sheaf of the Homology groups over X. In addition, $\bar{H}_{[H_c, H_c]}$ is called Homology covering space of X [8].

Then we state the following theorem.

Theorem 4.1. Let X be a connected complex manifold of dimension n with fundamental group $H_x \neq \{1\}$, $c \in X$ be an arbirarty fixed point and H_c be the fundamental group of X with respect to the point c. Then,

i) Each subgroup of H_c determines a sheaf over X which is a regular covering space of X.

ii) Each normal subgroup of H_c determines a sheaf of groups over X which is a regular covering space of X.

iii) Each normal subgroup of H_c that contain $[H_c,H_c]$ determines a sheaf of Abelian groups over X which is a regular covering space of X. Moreover the sheaf $\overline{H}_{[H_c,H_c]}$ of homology groups determined by $[H_c,H_c]$ is maximal and the proper normal subgroups of H_c that differently contain $[H_c,H_c]$ satisfy the descending chain condition.

Let H be the sheaf of the fundamental groups, $\Gamma(X,H)$ be the group of global sections of H over X and $N \subset H$ be a normal subsheaf. If $N \subset H$ is a normal subsheaf, then $\Gamma(X,N) \subset \Gamma(X,H)$ is a normal subgroup. Let N_c be the normal subgroup of H_c that corresponds to $\Gamma(X,N)$. Then, the groups $\Gamma(X,N)$ and N_c determine the sheaves Q_N and $\overline{H}_N$, respectively. Let us define a mapping Φ: $Q_N \rightarrow \overline{H}_N$ such that $\Phi((W, [s])_x) = \bar{s}(x)$, for any $(W, [s])_x \in Q_N$. It can be proved that Φ is a sheaf isomorphism. Hence, we identify $(W, [s])_x$ with $\bar{s}(x)$ and the section $\Upsilon[s]$ with $\bar{s}$ by means of Φ.

Therefore, theorem 3.7*. and theorem 3.1, are equivalent.

Let X be a connected complex analytic manifold of dimension n with fundamental group $\neq \{1\}$ and A(X) be the vector space (or C-Algebra) of holomorphic functions on X [5].

Let $f \in A(X)$ and $x \in X$ a point. f can be expanded into a power series f_x convergent at z, the local parameter of x. The totality of such power series at x as f runs through A(X) is denoted by A_x which is again a vector space (or C-Algebra) isomorphic to A(X). The disjoint union $A = \bigvee_{x \in X} A_x$ is a set over X with a natural projection π: $A \rightarrow X$ mapping each f_x onto the point of expansion x.

A natural topology on A was introduced *in* [5]. *In that topology* π is locally topological mapping. Hence (A, π) is a sheaf over X. The sheaf A is called the Restricted sheaf of germs of the totality of holomorphic functions A(X) on X [5]. In paper [6], it is shown that the Restricted sheaf A *is an analytic sheaf. Now, if* $I \subset A$ *is a restricted analytic subs*heaf, then $I_x \subset A_x$ is an ideal, for each $x \in X$. For this reason I is called a restricted ideal sheaf. Here the I_x's are isomorphic. On the other hand, in paper [7, 9], it is shown that;

i) The sheaf $\overline{H}_{[H_c,H_c]}$ of the homology groups over X is isomorphic to the Restricted sheaf A of germs of the totality of holomorphic functions A(X) on X.

ii) If $N_c \supset [H_c,H_c]$ is a normal subgroup of H_c and $\bar{H}_N$ is the cor responding sheaf, then $\bar{H}_N$ is isomorphic to a restricted ideal sheaf of A.

On the other hand, the sheaf $\bar{H}_{[H_c, H_c]}$ is isomorphic to the Quotient Sheaf $Q_{[H,H]}$ and the sheaf $\bar{H}_N$ is isomorphic to the Quotient Sheaf Q_N.

We then state the Fundamental Theorem [9] as follows:

Theorem 4.2. Let X be a connected complexanalytic manifold of dimension n with fundamental group $H_X \neq \{1\}$ and H be the sheaf of the fundamental groups over X. Then, the Commutator Subsheaf [H,H] of H determines the Restricted Sheaf A of germs of the totality of holomorphic functions A(X) on X. The normal subsheaves of H that contain [H,H] determine the restricted ideal sheaves of A.

Or equivalently,

Theorem 4.2. Let X be a connected complex analytic manifold of dimension n with fundamental group $H_X \neq \{1\}$, H be the sheaf of the fundamental groups over X and Γ(X,H) be the group of the global sections of H over X. Then, the Commutator Subgroup Γ(X, [H,H]) of Γ(X,H) determines the Restricted Sheaf A of germs of the totality of holomorphic functions A(X) on X. The normal subgroups of Γ(X,H) that contain Γ(X, [H,H]) determine the Restricted Ideal Sheaves of A.

From theorem 4.2.*., the Commutator Subgroup Γ(X, [H,H]) completely determines the vector space A(X) of holomorphic functions on X, because the Quotient Group Γ(X,H) /Γ(X, [H,H]) is isomorphic to the group A(X). Thus, each coset s. Γ(X, [H,H]), $s \in$ Γ(X,H), defines a holomorphic function on X. In other words, each coset s. Γ(X, [H,H]) defines a global section of the sheaf A over X.

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