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THE SHEAF OF THE GROUPS FORMED BY H-COGROUPS OVER TOPOLOGICAL SPACES

A.A. ÖÇAL - C. YILDIZ

SUMMARY

In this paper, we consider both homotopy and sheaf theory and construct an algebraic sheaf by means of the H-cogroups. Finally, we give some algebraic topological characterizations.

1. The Sheaf of the Groups formed by H-Cogroups over topological spaces.

Let's recall the following definition.

Definition 1.1. Let X, S be two topological spaces, and π: S $\rightarrow$ X be a locally topological map. Then the pair (S, π) or shortly S is called a sheaf over X.

Let $\mathcal{C}$ be the category of topological spaces X satisfying the property that all pointed topological spaces (X,x) with x $\in$ X have the same homotopy type. This category includes for example all topological vector spaces. Let us take x$\in\mathcal{C}$ as a base set if Q is any H-cogroup, then the set of homotopy class of homotop maps preserving the base points from (Q,q_0) to (X,x_i), i$\in$I, $[Q; (X,x_i)]$ obtained for each x$\in$X, (X,x) pointed topological spaces. i.e, $P(X) = \bigvee_{x\in X} [Q; (X,x)]$. Thus P(X) is a set over X. Let us now define a map Ψ : P(X) $\rightarrow$ X as follows; Let $\sigma\in$ P(X), then there exists x$\in$X such that $\sigma \in$ [Q:(X,x)]. Define Ψ $(\sigma) = x$ if $x_0 \in X$ is an arbitrarily fixed point, then let us denote by $W=W(x_0)$ open neighborhood of x_0 in X. Now, we can define a mapping s: W $\rightarrow$ P(X) as follows:

If $x_0 \in X$, then there exists a group $[Q: (X,x_0)]$ in P(X). Let $[f]x_0$ be a homotopy class in the group $[Q: (X,x_0)]$. If y is any point in W,

* Department of Mathematics, Faculty of Science and Arts, İnönü University, MALATYA.

then (X,x_0) and (X,y) are haveing the same homotopy type. Therefore, there is a homotopy equivalence map Φ: $(X,x_0) \to (X,y)$.

Hence

$$\begin{array}{ccc} (Q,q_0) & \xrightarrow{f} & (X,x_0) \\ & {\scriptstyle \Phi\, of} \searrow & \downarrow \Phi \\ & & (X,y) \end{array}$$

from diagram too, the map Φ of: $(Q,q_0) \to (X,y)$ is continuous and base-point preserving. $[h]_y \in [Q; (X,y)]$ is a homotopy class of map Φ of $= h$. Therefore, we define $s(y) = [h]_y$. In this way s is welldefined and

1. $(\Psi\, os)(y) = \Psi(s(y)) = y$, for each $y \in W$. Therefore $\Psi\, os = I_W$.

2. If, x_0 is an arbitrarily fixed point in W, $s(x_0) = [I_X of]_{x_0} = [f]_{x_0}$ for $W = W(x_0)$. Hence it can be written as $s(w) = \bigcup_{y \in W} [h]_y$.

If we can define s(w) as an open set, then it can be easilly shown that the

$$\zeta = \{s(W): W = W(x) \subset X,\ x \in X\}$$

family is a topology-base on P(X). Thus P(X) is a topological space.

Now, we can show that map Ψ is local topological. If, $\sigma = [h]_y \in P(X)$ and $y \in x$, then $\Psi(\sigma) = \Psi([h]_y) = y$. Therefore, there is a map s: $W \to P(X)$ such that $s(y) = \sigma$, $y \in W = W(x_0)$. Now, let us assume that $U(\sigma) = s(W)$ and $\Psi | U = \Psi^*$.

1. The map $\Psi^* = \Psi | U$: $U \to W$ is injective. Because for any σ_1, $\sigma_2 \in s(W)$ there are the points y_1, y_2 respectively in W such that $\sigma_1 = s(y_1) = [\Phi\, of]_{y1}$, $\sigma_2 = s(y_2) = [\Phi' of]_{y2}$. That is, we have the following diagrams:

$$\begin{array}{ccc} (Q,q_0) & \xrightarrow{f} & (X,x_0) \\ & {\scriptstyle \Phi\, of} \searrow & \downarrow \Phi \\ & & (X,y) \end{array} \qquad \begin{array}{ccc} (Q,q_0) & \xrightarrow{f} & (X,x_0) \\ & {\scriptstyle \Phi' of} \searrow & \downarrow \Phi' \\ & & (X,y) \end{array}$$

If, $\Psi^*(\sigma_1) = \Psi^*(\sigma_2)$, than $\Psi^*(s(y_1)) = \Psi^*(s(y_2)) \Rightarrow \Psi^*([\Phi of]_{y1}) = \Psi^*([\Phi' of]_{y2}) = y_1 = y_2$. Therefore $\Phi \sim \Phi' \Rightarrow \Phi$ of $\sim \Phi'$of $\Rightarrow [\Phi of]_{y1} = [\Phi' of]_{y2} = \sigma_1 = \sigma_2$.

2. The map $\Psi^* = \Psi|U: U \to W$ is continuous. In fact, if $\sigma \in U = s(W) \Rightarrow \Psi^*(\sigma) = y \in W$ and $V = V(y) \subset W$ is a neighborhood of y, then $s(V) \subset U = s(W)$ is neighborhood of σ and $\Psi^*(s(V)) = V \subset W$. So Ψ^* is continuous.

3. $\Psi^{*-1} = (\Psi \mid U)^{-1} = s: W \Rightarrow U = s(W)$ is continuous. In fact, if y is any point in W, $s(y) = \sigma \in U$ and $U' = U'(\sigma) \subset U$ is a neighborhoodof σ, then $(\Psi \mid U)(U') \subset W$ is a neighborhood of y in W and $s((\Psi|U)(U')) = U' \subset U$. So Ψ^{*-1} is continuous.

Hence, the following theorem can be given:

Theorem 1.1. Let Q be any H-cogroup and $X \in \mathcal{C}$. If $P(X) = \bigvee_{x \in X} [Q; (X,x)]$ and $\Psi: P(X) \to X$ such that $\Psi(\sigma) = \Psi([f]_x) = x$ for every $\sigma = [f]_x \in P(X)$, $x \in X$, then there is the natural topology over P(X) such that Ψ is locally topological with respect to this topology. Thus the pair $(P(X), \Psi)$, is a sheaf over X.

Definition 1.2. The sheaf $(P(X), \Psi)$ given by Theorem 1.1 is called the sheaf of the groups formed by Q, H-cogroup over (X,x) pointed topological spaces.

Definition 1.3. The group $[Q; (X,x)] = \Psi^{-1}(x)$ is called the stalk of the sheaf $(P(X), \Psi)$ over X and denoted by $P(X)_x$ for every $x \in X$.

Now, if $x \in X$ is an arbitrarily fixed point, and if W is open neighborhood of x in X, the mapping $s: W \to P(X)$ as defined in the construction of topology of P(X) is called a section of P(X) over W. Let us denote the collection of all sections of P(X) over W, by $\Gamma(W, P(X))$. It is easily shown that $\Gamma(W, P(X))$ is a group with respect to the following pointwise multiplication

$$(s_1.s_2)(y) = s_1(y)\, s_2(y),\ s_1, s_2 \in \Gamma(W, P(X)) \text{ and } y \in W$$

It follows from this definition that the operation of multiplication is well-defined and closed. Clearly, the operation of multiplication is associative and the mapping $I: W \to P(X)$ is the identity element which is obtained by means of the identity element of $[Q; (X,x)]$. On the other hand, the any inverse element of $s \in \Gamma(W, P(X))$, namely, $s^{-1} \in \Gamma(W, P(X))$ which is obtained by means of the homotopy inverse of H-cogroup Q. Hence $\Gamma(W, P(X))$ is a group. Thus the operation $(.): P(X) \oplus P(X) \to P(X)$ (That is, $(\sigma_1, \sigma_2) \to \sigma_1 . \sigma_2$ for every $\sigma_1, \sigma_2 \in P(X)$) is continuous. Hence $(P(X), \Psi)$ is algebraic sheaf.

2. The Characterizations.

Let Q be any H-cogroup and X_1, X_2 be two topological spaces in the category $\mathcal{C}$. Let $P(X_1)$, $P(X_2)$ be the corresponding sheaves respectively. Let us denote these as the pairs $(X_1, P(X_1))$ and $(X_2, P(X_2))$.

Definition 2.1. Let the pairs $(X_1, P(X_1))$ and $(X_2, P(X_2))$ be given. We say that there is a homomorphism between these pairs and write $F = (\beta^*, \beta): (X_1, P(X)_1)) \rightarrow (X_2, P(X_2))$, if there exists a pair $F = (\beta^*, \beta)$ such that

1. $\beta: X_1 \rightarrow X_2$ is a surjective and continuous mapping,
2. $\beta^*: P(X_1) \rightarrow P(X_2)$ is a continuous mapping,
3. β^* preserves the stalks with respect to β. That is, the following diagram is commutative.

$$\begin{array}{ccc} P(X_1) & \xrightarrow{\beta^*} & P(X_2) \\ \Psi_1 \downarrow & \beta & \downarrow \Psi_2 \\ X_1 & \longrightarrow & X_2 \end{array}$$

4. For every $x_1 \in X_1$ the restricted map $\beta^* \mid P(X_1)_{x_1}: P(X_1)_{x_1} \rightarrow P(X_2)_{\beta(x_1)}$ is a homomorphism.

Definition 2.2. Let the pairs $(X_1, P(X_1))$ and $(X_2, P(X_2))$ be given such that the map $F = (\beta^*, \beta): (X_1, P(X_1)) \rightarrow (X_2, P(X_2))$ is a homomorphism. Then the map $F = (\beta^*, \beta)$ is called an isomorphism and can be written $(X_1, P(X_1)) \overset{F}{\cong} (X_2, P(X_2))$, if the maps β^* and β are topological. Then the pairs $(X_1, P(X_1))$ and $(X_2, P(X_2))$ are called isomorphic.

Theorem 2.1. Let the pairs $(X_1, P(X_1))$ and $(X_2, P(X_2))$ be given. *If the map $\beta: X_1 \rightarrow X_2$ is given as surjective and continuous map, then* there exists a homomorphism between the pairs $(X_1, P(X_1))$ and $(X_2, P(X_2))$.

Proof: Let $x_1 \in X_1$ be can an arbitrarily fixed point. Then $\beta(x_1) \in X_2$ and $[Q; (X_1, x_1)] = P(X_1)_{x_1} \subset P(X_1)$, $[Q; (X_2, \beta(x_1))] = P(X_2)(x_1) \subset P(X_2)$ are the corresponding stalks.

If (X_1, x_1), $(X_2, \beta(x_1))$ are pointed topological spaces and f_1, g_1 are base-points preserving continuous maps from (Q, q_o) to (X_1, x_1), then f_2, g_2 base-points preserving continuous maps from (Q, q_o) to $(X_2, \beta(X_1))$ can be defined as $f_2 = \beta \circ f_1$, $g_2 = \beta \circ g_1$, respectively. Furthermore, if $f_1 \sim g_1$ rel. q_o, then it can be easily shown that $f_2 \sim g_2$ rel. q_o. Thus the correspondence $[f]_{x_1} \to [\beta o f]_{\beta(x_1)}$ is well-defined, and it maps homotopy classes of basepoints preserving continuous maps from (Q, q_o) to (X_1, x_1), to the homotopy classes of base-points preserving continuous maps from (Q, q_o) to $(X_2, \beta(x_1))$. That is, to each element $[f]_{x_1}$ there corresponds a unique element $[\beta\ o f]_{\beta(x_1)}$.

Since the point $x_1 \in X_1$ is arbitrarily fixed, the above correspondence gives us a map $\beta^*: P(X_1) \to P(X_2)$ such that $\beta^*([f]) = [\beta o f] \in P(X_2)$, for every $[f] \in P(X_1)$.

$$\begin{array}{ccc} P(X_1) & \xrightarrow{\beta^*} & P(X_2) \\ \Psi_1 \downarrow & & \downarrow \Psi_2 \\ X_1 & \xrightarrow{\beta} & X_2 \end{array}$$

1. β^* is a continuous. Because if $U_2 \subset P(X_2)$ is any open set, then it can be shown that $\beta^{*-1}(U_2) = U_1 \subset P(X_1)$ is an open set. In fact, if $U_2 \subset P(X_2)$ is an open set, then $U_2 = \bigcup_{i \in I} s^2{}_i (W_i)$ and $\Psi_2(U_2) = \bigcup_{i \in I} W_i$, where the W_i's are open neighborhoods and the $s^2{}_i$'s are sections over W_i. Thus, $\bigcup_{i \in I} W_i \subset X_2$ is an open set and $\beta^{-1}(\bigcup_{i \in I} W_i) = \bigcup_{i \in I} \beta^{-1}(W_i) \subset X_1$ is an open set since β is a surjective and continuous map. Furthermore, since $\beta^{-1}(W_i)$, $i \in I$ are open neighborhoods in X_1, there exists sections $s^1{}_i: \beta^{-1}(W_i) \to P(X_1)$ such that $\bigcup_{i \in I} s^1{}_i (\beta^{-1}(W_i)) \subset P(X_1)$ is an open set. Let us now show that $U_1 = \bigcup_{i \in I} s^1{}_i (\beta^{-1}(W_i))$.

If $\sigma_1 = [f]_{x_1} \in U_1$ is any element, then there exists a $\sigma_2 = [\beta o f]_{\beta(x_1)} \in U_2$ such that $\beta^*(\sigma_1) = \sigma_2$ and $\Psi_2(\sigma_2) = \Psi_2([\beta o f]_{\beta((x_1))}) = \beta(x_1) = x_2$. Hence, if $\beta(x_1) = x_2 \in W_i$ for at least one $i \in I$, then $x_1 \in \beta^{-1}(W_i)$ and $\sigma_1 = [f]_{x_1}$ $\bigcup_{i \in I} s^1{}_i (\beta^{-1}(W_i))$. Hence $U_1 \subset \bigcup_{i \in I} s^1{}_i (\beta^{-1}(W_i))$. On the other hand $\sigma_1 \in \bigcup_{i \in I} s^1{}_i (\beta^{-1}(W_i))$ implies that $\sigma_1 \in s^1{}_i (\beta^{-1}(W_i))$ for at least one $i \in I$. from here if $\sigma_1 = [f]_{x_1}$, then $\Psi_1(\sigma_1) = x_1$ and $\beta o f$ is a base-point preserving continuous map from (Q, q_o) to (X_2, x_2), where $\beta(x_1) = x_2 \in W_i$. Thus $[\beta o f]_{x_2} = \sigma_2 \in U_2$. Hence $\sigma_1 \in U_1$ and $\bigcup_{i \in I} s^1{}_i (\beta^{-1}(W_i)) \subset U_1$. Therefore, $U_1 = \bigcup_{i \in I} s^1{}_i\ \beta^{-1}(W_i))$. Hence β^* is a continuous map.

2. β^* is preserves the stalks with respect to β. In fact, for any $\sigma_1 = [f]_{x_1} \in P(X_1)_{x_1} \subset P(X_1)$

$$(\beta \text{ o } \Psi_1)\ ([f]_{x_1}) = \beta\ (\Psi_1 ([f]_{x_1})) = \beta(x_1) = x_2.$$

$$(\Psi_2 \text{ o } \beta^*)\ ([f]_{x_1}) = \Psi_2\ (\beta^* ([f]_{x_1})) = \Psi_2\ ([\beta \text{ of }]_{x_2}) = x_2.$$

3. For every $x_1 \in X_1$ the map $\beta^* \mid P(X_1)_{x_1} : P(X_1)_{x_1} \to P(X_2)_{(x_1)}$ is a homomorphism. In fact, if the maps f_1, g_1 are the base-point preserving continuous maps from (Q,q_0) to (X_1,x_1) for $x_1\ X_1$ and $f_2 = \beta o f_1$, $g_2 = \beta o g_1$: $(Q,q_0) \to (X_2,\ \beta(x_1))$ are the corresponding maps, then $[f]_{x_1}$, $[g]_{x_1} \in P(X_1)_{x_1}$ and $[\beta of]_{\beta(x_1)}$, $[\beta og]_{\beta(x_1)} \in P(X_2)_{\beta(x_1)}$.

Now, if $[f]_{x_1}$, $[g]_{x_1} \in P(X_1)_{x_1}$, then

$$\begin{aligned} \beta^* ([f]_{x_1}\ [g]_{x_1}) &= \beta^*([(f,g)o\nu]_{x_1}) = [\beta o\ (f,g)o\nu]_{\beta(x_1)} \\ &= [(\beta of,\ \beta og)\ o\nu]_{\beta(x_1)} \\ &= [\beta of]_{\beta(x_1)}\ [\beta og]_{\beta(x_1)} \\ &= \beta^* ([f]_{x_1})\ \beta^* ([g]_{x_1}), \end{aligned}$$

where ν is the multiplication of H-cogroup Q.

Thus $F = (\beta^*,\beta)$ is a homomorphism.

Theorem 2.2. Let the pairs $(X_1, P(X_1))$, $(X_2, P(X_2))$, $(X_3, P(X_3))$ and surjective and continuous maps $\beta_1: X_1 \to X_2$, $\beta_2: X_2 \to X_3$ begiven. Then, there exists a homomorphism $F = (\beta^*, \beta): (X_1, P(X_1)) \to (X_3, P(X_3))$ such that $\beta = \beta_2 \text{ o } \beta_1$, $\beta^* = \beta^*_2 \text{ o } \beta^*_1$.

Proof: Since $\beta_2 \text{ o } \beta_1: X_1 \to X_3$ is a surjective and continuous map, there exists a homomorphism $F = (\beta^*, \beta): (X_1, P(X_1)) \to (X_3, P(X_3))$ (Theoıem 2.1). To prove this theorem it is sufficient to show that $\beta^* = \beta^*_2 \text{ o } \beta^*_1$. In fact, for any $[f] \in P(X_1)$, we must show that $\beta^* ([f]) = (\beta^*_2 \text{ o } \beta^*_1)\ ([f])$. However

$$\begin{aligned} \beta^* ([f]) = [\beta of] = [(\beta_2 \text{ o } \beta_1) \text{ of}] &= [\beta_2 \text{ o } (\beta_1 \text{ of})] \\ &= \beta^*_2 ([\beta_1 \text{ of}]) = \beta^*_2 (\beta^*_1 ([f])) \\ &= (\beta^*_2 \text{ o } \beta^*_1) ([f]). \end{aligned}$$

Therefore $\beta^* = \beta^*_2 \text{ o } \beta^*_1$.

Now, we can state the following theorem:

Theorem 2.3. There is a covariant functor from the category $\mathcal{C}$ and surjective continuous maps to the category of sheaves and sheaf homomorphisms.

Theorem 2.4. Let the pairs $(X_1, P(X_1))$ and $(X_2, P(X_2))$ be given. If the map $\beta: X_1 \rightarrow X_2$ is a topological map, then there exists an isomorphism between the pairs $(X_1, P(x_1))$ and $(X_2, P(X_2))$.

Proof: By Theorem 2.1, there exists a homomorphismF $= (\beta^*, \beta)$ between the pairs $(X_1, P(X_1))$ and $(X_2, P(X_2))$. To prove this theorem it is sufficient to show that β^* is one-to-one and β^{*-1} is continuous.

Since β is a topological map it has a continuous inverse β^{-1}. So, by Theorem 2.1 there exists homomorphism $F = (\beta^*, \beta)$ and $F = ((\beta^{-1})^*, \beta^{-1})$. On the other hand, for any two elements $[f_1], [g_1] \in P(X_1)$, $\beta^*([f_1]) = \beta^*([g_1])$ implies $[f_2] = [g_2] \Rightarrow (\beta^{-1})^*([f_2]) = (\beta^{-1})^*([g_2])$. Thus, $(\beta^{-1})^*(\beta^*([f_1])) = (\beta^{-1})^*(\beta^*([g_1]))$. Since $(\beta^{-1})^* \circ \beta^* = (\beta^{-1} \circ \beta^*)$ and $\beta^{-1} \circ \beta = I_{X_1}, (\beta^{-1} \circ \beta)^* = I_{P(x_1)}$ and $[f_1] = [g_1]$. Hence β^* is one-one. Since $\beta^{*-1} = (\beta^{-1})^*$, β^{*-1} is continuous.

Therefore, $F = (\beta^*, \beta)$ is an isomorphism.

ÖZET

Bu makalede, bir Q H-cogrubu vasıtasıyla bir cebirsel yapılı demet oluşturulmuş ve bazı cebirsel topolojik karakterizasyonlar verilmiştir.

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