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THE 3- PLANE AND THE LIGHT CONE

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ABSTRACT

In this paper we show that the 3-plane passing through the origin in a space-time will intersect the light cone in two perpendicular 2-planes.

1- The Principal Planes:

In this section we will give a sketch of how the principal planes can be obtained in order to be able to discuss the way in which a 3- plane intersect the light cone in space time.

A principal plane is a diametral plane which is at right angles to the chords which it bisects. Now if the axes are rectangular, the diametral plane (whose equation is

$$\iota \frac{\partial F}{\partial x} + m \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} = 0, \text{ where}$$

$$F(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0$$
or $x(a\iota + hm + gn) + y(h\iota + bm + fn) + z(g\iota + fm + cn) = 0)$

is at right angles to the line $\dfrac{x}{\iota} = \dfrac{y}{m} = \dfrac{z}{n}$, if

$$\frac{a\iota + hm + gn}{\iota} = \frac{h\iota + bm + fn}{m} = \frac{g\iota + fm + cn}{n}$$

If each of these ratios is equal to λ, then

$$\begin{aligned}(a-\lambda)\,\iota + hm + gn &= 0,\\ h\,\iota + (b-\lambda)\,m + f\,n &= 0, \qquad \text{(i)}\\ g\,\iota + fm + (c-\lambda)\,n &= 0\end{aligned}$$

Therefore, λ is a root of the equation:

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

or equivalently,

$$\lambda^3 - \lambda^2(a+b+c) + \lambda(bc+ca+ab-h^2-g^2-f^2) - D = 0 \qquad \text{(ii)}$$

where

$$D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Equation (ii) is called the discriminating cubic. It gives three values of λ, to each of which corresponds a set of values for (ι, m, n) and by substituting these sets in the equation $\iota \frac{\partial F}{\partial x} + m \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} = 0$, which by means of the relations (i) reduces to $\lambda(\iota x + my + nz) = 0$, we obtain the equations of three principal planes [1].

2– The Main Result:

A 3-plane in space time has the equation $\Sigma A_r x_r + B = 0$, $r = 1,2,3,4$. This equation will reduce to

$$\Sigma A_r x_r = 0,\ r = 1, 2, 3, 4, \qquad (1)$$

if the 3-plane is passing through the origin. In this case it will have a unique orthogonal line through the origin with equations $x_r = A_r u$, where u is a parameter.

Now consider the equation of the light cone:

$$\sum_{i=1}^{3} x_i^2 - x_4^2 = 0 = \langle x,x \rangle = 0 \qquad (2)$$

and rewrite equation (1) as follows:

$$\sum_{i=1}^{3} B_i x_i + x_4 = 0,\ B_i = A_i / A_4 \qquad (1)'$$

From (1)' and (2) we have

$$\Big(\sum_{i=1}^{3} B_i x_i\Big)^2 = (-x_4)^2 = x_4^2 = \sum_{i=1} x_i^2$$

$$\therefore \sum_{i=1}^{3} B^2_i x^2_i + 2 \sum_{\substack{i,j=1\\ i\neq j}}^{3} B_i B_j x_i x_j = \sum_{i=1}^{3} x^2_i$$

or

$$\sum_{i=1}^{3} C_i^2 x^2_i + 2 \sum_{\substack{i,j=1\\ i\neq j}}^{3} B_i B_j x_i x_j = 0,\ C^2_i = B^2_i - 1 \qquad (3)$$

The discriminating cubic for equation (3) is:

$$\lambda^3 - \lambda^2 (C^2_1 + C^2_2 + C^2_3) + \lambda (C^2_2 C^2_3 + C^2_1 C^2_3 + C^2_1 C^2_2 -$$

$$- B^2_2 B^2_3 - B^2_1 B^2_3 - B^2_1 B^2_2) - D = 0,$$

where

$$D \equiv \begin{vmatrix} C^2_1 & B_1B_2 & B_1B_3 \\ B_2B_1 & C^2_2 & B_2B_3 \\ B_1B_3 & B_2B_3 & C^2_3 \end{vmatrix} = C^2_1 + C^2_2 + C^2_3 + 2 \neq 0$$

Thus: $\lambda^3 - A\ \lambda^2 - (2A + 3)\lambda - (A + 2) = 0$, $A \equiv C^2_1 + C^2_2 + C^2_3$, and so, $(\lambda + 1)^2 [\lambda - (A+2)] = 0$. It follows that $\lambda = -1, -1, A+2$.

Consider first $\lambda = -1$. In this case the set of equations (i) may reduce to the single equation:

$$B_1 \iota_1 + B_2 m_1 + B_3 n_1 = 0,\ i = 1, 2, 3, \qquad (4)$$

If we consider $\lambda = A+2$, the set of equations (i) may take the from:

$$\begin{aligned} (B^2_2 + B^2_3)\, \iota_3 + B_1B_2 m_3 + B_1B_3\, n_3 &= 0, \\ B_1B_2 \iota_3 - (B^2_1 + B^2_2)\, m_3 + B_2B_3 n_3 &= 0, \\ B_1B_3 \iota_3 + B_2B_3 m_3 - (B^2_1 + B^2_2)\, n_3 &= 0 \end{aligned} \qquad (5)$$

Dividing the first equation of (5) by B_1 and the second by B_2 then substracting, we have:

$$\frac{\iota_3}{B_1} = \frac{m_3}{B_2}$$

Again from the second and third equations of (5), we get

$$\frac{m_3}{B_2} = \frac{n_3}{B_3}$$

Thus,

$$\frac{\iota_3}{B_1} = \frac{m_3}{B_2} + \frac{n_3}{B_3} \tag{6}$$

From the above results we find that the single equation (4) corresponding to $\lambda_1 = \lambda_2 = -1$, is the condition that the directions given by (ι_1, m_1, n_1) and (ι_3, m_3, n_3) should be at right angles. The principal planes corresponding to the directions (ι_1, m_1, n_1) and (ι_3, m_3, n_3) are respectively: $\iota_1 x_1 + m_1 x_2 + n_1 x_3 = 0$ and $\iota_3 x_1 + m_3 x_2 + n_3 x_3 = 0$, or equivalently: $\iota_1 x_1 + m_1 x_2 + n_1 x_3 = 0$ and $B_1 x_1 + B_2 x_2 + B_3 x_3 = 0$, where $B_1 \iota_1 + B_2 m_1 + B_3 n_1 = 0$. It follows that the 3-plane which pass through the origin will intersect the light cone in two perpendicular planes.

REFERENCE

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