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On Finiteness of Homological Dimension of Generalized Homology Modules of Finite-Dimensional CW-Complexes

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On Finiteness of Homological Dimension of Generalized Homology Modules of Finite-Dimensional CW-Complexes

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ABSTRACT

The finiteness of homological dimension of complex bordism and cobordism modules of finite CW-complexes was proved by Adams [1] and Conner-Smith [5]. It was proved by Koçak [7] that homological dimensions of complex bordism modules of finite-dimensional CW-complexes are still finite. We now give some conditions for the finiteness of homological dimension of generalized homology modules of finite - dimensional CW-complexes. In the case of finite CW-complexes the coherence of the coefficient ring and one additional condition (again on the coefficient ring alone) imply the finiteness of homological dimension of generalized homology modules.

1. INTRODUCTION

It was shown by Novikov and Smith ([10] that the complex bordism and cobordism modules ($MU_*(X)$ and $MU^*(X)$) of finite CW-complexes are finitely generated over the coefficient rings (MU_* and MU^*). Conner-Smith [5] and Adams [1] proved the finiteness of the homological dimension of these modules. (For complex (co-)bordism see Conner-Floyd [3] and Adams [2], for homological dimension see MacLane [8]). As an example of geometric significance of this algebraic property the following theorem can be given: For a finite CW-complex X, an integral homology class in X is representable by a weakly almost-complex manifold (i.e. the Thom-homomorphism $MU_*(X) \to H_*(X)$ is surjective) if and only if the homological dimension of $MU_*(X)$ over MU_* is 0 or 1 (≤ 1). (Smith [11], Conner-Smith [5]).

Using a result of Quillen [9] on dimensions of generators of $MU_*(X)$ for a finite CW-complex X (there are generators of degree $\leq 2n$ if X has dimension n) we proved in [7] that the homological dimension of $MU_*(X)$

is still finite for a finite-dimensional (but possibly infinite) CW-complex X. Now we will formulate a theorem on finiteness of homological dimension of generalized homology modules of finite-dimensional CW-complexes under conditions, which seem to be a reasonable generalisation of the complex bordism case. Coherence of the coefficient ring is not needed but is replaced by a condition what we call "semi-polynomiality". Finite generation of the modules is replaced by the "boundedness above of degrees of a set of generators".

Specializing to the case of finite CW-complexes we show the finiteness of homological dimension of generalized homology modules for a coherent and semi-polynomial coefficient ring. This result could be extracted from the general setting of Adams [1] but although he treats the finite generation of generalized homology modules of finite CW-complexes for an arbitrary theory, he establishes the theorem on finiteness of homological dimension for complex cobordism alone.

2. HOMOLOGICAL DIMENSION AND A FINITENESS THEOREM.

We briefly recall some definitions, then single out a class of graded rings which we call semi-polynomial rings and prove a theorem on finiteness of homological dimension of some modules over semi-polynomial rings as a generalization of a theorem of Conner-Smith [5], which is also related to a theorem of Adams [1].

2.1. Let R be a ring with unit and M a (left) module over R (Modules will be left modules unless otherwise stated). If there is an exact sequence (a resolution)

$$0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0,$$

where P_i $(i=0,1,\ldots,n)$ are projective modules over R, then it is said that the homological (or projective) dimension of M over R is less than or equal to n: $\text{hom.dim}_R\ M \leq n$.

The least upper bound in this sense is called the homological dimension of M over R. If for all modules over a fixed ring R homological dimensions are bounded with n ($\text{hom. dim}_R\ M \leq n$ for all R-modules M), then it is said that the (left) global dimension of R is less than or equal to n. The least upper bound in this sense, or in other words, the maximum of homological dimensions of R-modules M., is called the (left) global dimension of R. If there is a module M over R not allowing a finite projective resolution, then the (left) global dimension of R is said to be infinite.

Examples: Global dimension of a field is zero, glob. dim $Z = 1$. Global dimension of a polynomial ring $Z[x_1, x_2, \ldots, x_n]$ is $n+1$. (A version of the Hilbert syzygy theorem).

With the help of the functor Ext the condition $\text{hom.dim}_R M \leq n$ can be characterized by the property that for all modules N over R Ext $\text{Ext}^{n+1}(M,N)$ vanishes. The long exact sequence of Ext enables one to show easily the following

Lemma: Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If two of them have finite homological dimension, so has the third.

2.2. To show the finiteness of the homological dimension of the complex (co-)bordism modules of finite CW-complexes Conner-Smith) [5] proved the following

Theorem : Let $R = Z[x_1, x_2, x_3, \ldots]$ (the polynomial ring in the variables $x_1, x_2, x_3, \ldots$), F be a free R-module and $M \subset F$ be a finitely generated submodule. Then the $\text{hom.dim}_R M$ is finite.

We shall now generalize this theorem to the case where M is allowed to be non-finitely generated and R will be what we shall call a semi-polynomial ring:

Definition : Let $R = \{R_i\}_{i=0,1,2,\ldots}$ be a non-negatively graded ring and let R(n) denote the subring generated by elements of degree $\leq n$. If the rings R(n) have finite global dimensions and R is free as a right-module over the subring R(n) (for all n), then R will be called a semi-polynomial graded ring.

Remark: The polynomial coefficient rings of unoriented (co-)bordism and complex (co-)bordism are semi-polynomial.

We now formulate the generalization of the theorem above:

Theorem: Let R be semi-polynomial graded ring, F a free graded R–module and $M \subset F$ a submodule. Suppose that F has a basis of (homogeneous) elements with degrees bounded below and M has a generating set of elements with degrees bounded above. Then, $\text{hom.dim}_R M$ is finite.

Proof: Let S denote a basis of F and T a generating set of M with properties stated above . Every $t \in T$ can be expressed as $t = \Sigma r_k s_k$ with $r_k \in R$ and $s_k \in S$. Since the degrees of the elements s_k are bounded below, the degrees of r_k must be bounded above and thus there is a sub ring R(n) containing all elements appearing as coefficients in the expression of $t \in T$. Let $F(n) \subset F$ be the free R(n)–module with the same basis S. According to our choise is $T \subset F(n)$. The elements of T generate in the R (n)-module F (n) an R (n)-submodule, which we denote by M(n).

The isomorphism $R \underset{R(n)}{\otimes} R(n) \xrightarrow{\cong} R$ induces the isomorphism

$R \underset{R(n)}{\otimes} F(n) \xrightarrow{\cong} F$, which supplies us with the isomorphism

$R \underset{R(n)}{\otimes} M(n) \xrightarrow{\cong} M$. (The epimorphism is clear by $1 \otimes t \longrightarrow t$ and the monomorphism is a consequence of the monomorphism of $R \underset{R(n)}{\otimes} M(n) \longrightarrow R \underset{R(n)}{\otimes} F(n)$ and the latter is true because R is free on R(n)).

Since R(n) has finite global dimension, there is a finite resolution of M(n) consisting of projective R(n) - modules P_i:

$$0 \to P_m \to P_{m-1} \to \dots \to P_1 \to P_0 \to M(n) \to 0.$$

Since R is free on R(n) ,we obtain the following exact sequence by tensoring:

$$0 \to R \underset{R(n)}{\otimes} P_m \to R \underset{R(n)}{\otimes} P_{m-1} \to \dots \to R \underset{R(n)}{\otimes} P_0 \to R \underset{R(n)}{\otimes} M(n) \cong M \longrightarrow 0.$$

The terms of this sequence are naturally R-modules and they are projective: Since P_i is a projective R(n)-module, there is an R(n)-module Q_i, such that $P_i \oplus Q_i \cong \oplus R(n)$. We get

$$R \underset{R(n)}{\otimes} (P_i \oplus Q_i) \cong R \underset{R(n)}{\otimes} (\oplus R(n)) \cong \oplus (R \underset{R(n)}{\otimes} R(n)) \cong \oplus R$$

and $(R \otimes_{R(n)} P_i) \oplus (R \otimes_{R(n)} Q_i) \cong \oplus R$. This means that the R–module $R \otimes_{R(n)} P_i$ is projective. Hence $\text{hom.dim}_R M$ is finite.

Remark 1: It is clear from the proof that an upper bound for $\text{hom.dim}_R M$ can be found in the following way: Find a subring R(n) containing all coefficients of a set of generating elements (with degrees bounded above) of M with respect to a basis of F (with degrees bounded below). Then glob. dim R(n) is an upper bound for $\text{hom.dim}_R M$.

Remark 2: The proof reveals that it is enough to have a sequence of subrings $R(n_1)$, $R(n_2)$, ..., $R(n_i)$, ... ($n_i \to \infty$ for $i \to \infty$), such that $R(n_i)$ has finite global dimension and R is free as a right $R(n_i)$-module ($i=1, 2, ...$).

Remark 3: The condition that F has a basis of elements with degrees bounded below is automatically fulfilled for nicht-negatively graded free modules or for modules with a finite basis. The same is true for the second condition on M, if M is finitely generated.

3. HOMOLOGICAL DIMENSION OF GENERALIZED HOMOLOGY MODULES.

We shall now apply the last theorem to the modules of generalized homology theories. As is well-known, these are homology theories satisfying all of the Eilenberg-Steenrod axioms but the last dimension axiom (see Dold [6], Switzer [12], Conner-Floyd [4]). These theories can be defined on various categories but we assume the category of all topological pairs and continuous maps between pairs to be the domain category, since this is rather natural for bordism homology theories. The category of abelian groups will be the range category and we assume the homology theory to be strongly additive. Furthermore, the theory is supposed to be multiplicative. Since this is less standardized, we recall the properties of the multiplication: Let $h = \{h_p\}_{p \in Z}$ be a generalized homology theory and suppose that a collection of homomorphisms

$$h_p(X,A) \otimes h_q(Y,B) \to h_{p+q}(XxY, AxY \cup XxB) \quad (p,q \in Z)$$

is given for all pairs (X,A), (Y,B). For $\alpha \in h_p(X,A)$, $\beta \in h_q(Y,B)$ let us denote the image of $\alpha \otimes \beta$ by $\alpha \times \beta$. If the following conditions are satisfied, we call the theory with such a multiplication a multiplicative theory:

a) Naturality: For $f : (X,A) \to (X',A')$, $g : (Y,B) \to (Y',B')$ the following diagramm must commute:

$$\begin{array}{ccc} h_p(X,A) \otimes h_q(Y,B) & \xrightarrow{x} & h_{p+q}(XxY,\ AxY \cup XxB) \\ \downarrow f_* \otimes g_* & & \downarrow (fxg)_* \\ h_p(X',A') \otimes h_q(Y',B') & \xrightarrow{x} & h_{p+q}(X'xY', A'xY' \cup X'xB') \end{array}$$

that is: $(fxg)_* (\alpha x \beta) = (f_* \alpha) x (g_* \beta)$.

b) Associativity: For $\alpha \in h_p(X,A)$, $\beta \in h_q(Y,B)$, $\gamma \in h_r(Z,C)$ it holds $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$.

c) Commutativity: For $\alpha \in h_p(X,A)$, $\beta \in h_q(Y,B)$ and

$t: XxY \to YxX$, $t(x,y) = (y,x)$ it holds

$t_* (\alpha \times \beta) = (-1)^{pq} (\beta \times \alpha)$.

d) Unit element: There is an element $1 \in h_0(P)$ (P = the one-point space), such that $\alpha x 1 = 1 x \alpha$ for $\alpha \in h_p(X,A)$.

e) Stability: For $\alpha \in h_p(X,A)$, $\beta \in h_q(Y,B)$,

$i_1: (AxY, AxB) \to (AxY \cup XxB, AxB)$ and $i_2: (XxB, AxB) \to (AxY \cup XxB, AxB)$ it holds $\partial_{p+q}(\alpha \times \beta) = i_{1*}(\partial_p \alpha \times \beta) + i_{2*}((-1)^p \alpha \times \partial_q \beta)$. (For $B = \varnothing$ this reads $\partial_{p+q}(\alpha \times \beta) = \partial_p \alpha \times \beta$).

For $X = Y = P$ and $A = B = \varnothing$ this multiplication converts the graded abelian group $h_* = \{h_p(P)\}_{p \in Z}$ into a graded, commutative (in the graded sense) ring with unit; the graded abelian groups $h_*(X)$ and $h_*(X,A)$ into graded (left and right) h_*–modules; and the Z–homomorphisms $h_*(X,A) \to h_*(Y,B)$ and $h_*(X,A) \to h_*(A)$ into h_*–homomorphisms.

3.1. Homological Dimension of Generalized Homology Modules of Finite-dimensional CW-Complexes.

Let h be a non-negative ($h_p(X, A) = 0$ for $p < 0$), strongly additive multiplicative generalized homology theory and let the coefficient ring h_* be a semi-polynomial ring. We need one additional condition to establish the finiteness of $\text{hom.dim}_{h_*} h_*(X)$ for a finite-dimensional CW-complex X. It was proven by Quillen [9] that the complex bordism module $MU_*(X)$ of a finite CW-complex X has generators with degrees $\leq$ 2n, if X has dimension $\leq$ n and this was used in Koçak [7] to prove the finiteness of $\text{hom.dim}_{MU_*} MU_* (X)$ for a finite-dimensional CW-complex X.

We can formulate the above boundedness property of degrees of generators for a generalized homology theory in various ways:

Definition: Let h be a generalized homology theory. We say, h is bounded on finite CW-complexes (resp. on finite-dimensional CW-complexes) if $h_* (X)$ has generators with bounded degrees for finite CW-complexes (resp, for finite - dimensional CW-complexes).

We say ,h is uniformly bounded on finite CW-complexes (resp. on finite-dimensional CW-complexes) if there are generators of $h_* (X)$ with a fixed upper bound for all finite CW-complexes X with a fixed dimension (resp. for finite-dim. CW-complexes).

In the uniformly bounded case, there is a bounding function $\rho(n)$.

Example: Complex bordism is uniformly bounded on both finite and finite -dimensional CW-complexes with $\rho(n)=2n$.

Remark: Let h be a homology theory with the property that every element $\alpha \in h_*(X)$ for a CW-complex X comes from a compact subspace: That means, there is a finite subcomplex $Y \subset X$, such that α is in the image of $h_* (Y) \rightarrow h_* (X)$. (e.g. singular homology and all bordism theories possess this property). Now, if h is uniformly bounded on finite CW-complexes, then h is uniformly bounded on finite-dimensional CW complexes. For, let X be a finite -dimensional CW-complex and choose for every finite subcomplex $Y \subset X$ a set of generators and take the images of all of them by $h_*(Y) \rightarrow h_*(X)$. One sees that it is possible to choose the same bounding function.

Now we can formulate the following theorem.

Theorem: Let h be a non -negative, strongly additive, multiplicative generalized homology theory, which is bounded on finite-dimensional CW-complexes. If the coefficient ring h_* is semi-polynomial, then $\text{hom.dim}_{h_*} h_* (X)$ is finite for a finite-dimensional CW-complex X.

If h is uniformly bounded on finite-dimensional CW-complexes (cf. the last remark), then a fixed bound can be given in dependence of the dimension of X.

Proof: We prove the theorem by induction on the dimension of X. If X is zero-dimensional, so it is discrete and (by strong additivity) $h_*(X)$ is free and has homological dimension zero.

Suppose that the theorem is proven for all finite-dimensional CW-complexes with dimension $\leq$ n–1. We show that for an n-dimensional CW–complex X hom $.\dim_{h*} h_*(X)$ is finite.

Let X^{n-1} denote the (n–1)-skelett of $X = X^n$ and consider the exact triangle

$$\begin{array}{ccc} h_*(X^{n-1}) & \xrightarrow{i_*} & h_*(X^n) \\ & \partial_* \nwarrow \quad \swarrow j_* & \\ & h_*(X^n, X^{n-1}) & \end{array}$$

(i_* and j_* are induced by $X^{n-1} \xrightarrow{i} X^n$, $X^n \xrightarrow{j} (X^n, X^{n-1})$ and ∂_* is the boundary operator).

We can excise the following exact sequences from this exact triangle:

(1) $0 \to j_* h_*(X^n)) \to h_*(X^n, X^{n-1}) \to \partial_*(h_*(X^n, X^{n-1})) \to 0$

(2) $0 \to \partial_*(h_*(X^n, X^{n-1})) \to h_*(X^{n-1}) \to i_*(h_*(X^{n-1})) \to 0$

(3) $0 \to i_*(h_*(X^{n-1})) \to h_*(X^n) \to j_*(h_*(X^n)) \to 0.$

Now, $h_*(X^n, X^{n-1})$ is a free h_*-module (by suitable excision and strong additivity $h_*(X^n, X^{n-1}) \cong \bigoplus h_*(R^n, R^n - \{0\})$). Since by assumption $h_*(X^n)$ has a set of generators with bounded degrees, this is also true for $j_*(h_*(X^n))$. (Take the images of the generators by j_*). By applying the theorem of 2.2. to the submodule $j_*(h_*(X^n))$ of the free h_*–module $h_*(X^n, X^{n-1})$ we see that $\text{hom.dim}_{h*} j_*(h_*(X^n))$ is finite. By the lemma of 2.1. and the exact sequence (1) we get that $\text{hom.dim}_{h*}\partial_*(h_*(X^n, X^{n-1}))$ is finite. The induction hypothesis, the exact sequence (2) and lemma 2.1. gives then the finiteness of hom.dim h_* $i_*(h_*(X^{n-1}))$. Finally, again lemma 2.1., finiteness of hom. $\dim_{h*} i_*(h_*(X^{n-1}))$ and of $\text{hom.dim}_{h*} j_*(h_*(X^n))$ shows that $\text{hom.dim}_{h*}(h_*(X^n))$ is finite.

To prove the second part of the theorem, one can first easily improve the lemma 2.1. with the help of the long exact sequence of Ext to give estimations for homological dimension of each of the three modules in terms of homological dimensions of the other two. Combining this with the remark 1 to theorem 2.2. one can get by induction an upper bound for the $\mathrm{hom.dim}_{h_*}(h_*(X^n))$ depending only on the dimension n. But the computation of a precise value for an upper bound is now omitted.

Remark 1: For an individual pair (X,A) ($A \subset X$) of topological spaces the above theorem could be formulated in the following way: Let h be a non-negative, multiplicative generalized homology theory and let h_* be semi-polynomial. If $h_*(A)$ has finite homological dimension, $h_*(X)$ has generators bounded above (e.g. finitely generated) and $h_*(X,A)$ is a free h_*-module, then $\mathrm{hom.dim}_{n*}\, h_*(X)$ is finite. This is seen as above starting with the exact triangle

$$\begin{array}{ccc} h_*(A) & \xrightarrow{i_*} & h_*(X) \\ & \partial_* \nwarrow \quad \swarrow j_* & \\ & h_*(X,A) & \end{array}$$

Remark 2: The assumption of non-negativity of the homology theory is only made for convenience and can easily be removed by defining the semi-polynomiality for any graded ring and making all subsequent boundedness assumptions two-sided.

Remark 3: In the case of finite CW–complexes the boundedness condition should be replaced by the more natural finite generation. This is known to be true for a multiplicative (co-) homology theory with a coherent coefficient ring (Adams [1], Conner-Smith [5]). Thus the above thorem implies the following one:

Theorem: Let h be a (non-negative) multiplicative generalized homology theory and suppose that the coefficient ring h_* is coherent and semi-polynomial. Then $\mathrm{hom.dim}_{n*}\, h_*(X)$ is finite for a finite CW-complex X. If there is a fixed upper bound for generators for all CW–complexes of a fixed dimension, then a fixed upper bound for $\mathrm{hom.\ dim}_{n*}\, h_*(X)$ can be given depending on the dimension.

Remark: This theorem can be stated for general cohomology as well, whereas the former theorem cannot be carried over directly because of the freeness - question of $h_*(X^n, X^{n-1})$.

ÖZET

Sonlu CW- Komplekslerinin kompleks bordizm ve kobordizm modüllerinin homolojik boyutlarının sonluluğu Adams [1] ve Conner-Smith [5] tarafından gösterilmiş ve [7]' de kompleks bordizm modüllerinin homolojik boyutlarının sonlu-boyulu CW-kompleksleri için de sonlu olduğu ispatlanmıştı.

Burada sonlu-boyutlu CW-komplekslerinin genel homoloji modüllerinin homolojik boyutlarının sonluğu için bazı şartlar veriyoruz.

Sonlu CW-kompleksleri için, temel halkanın koherentiliği ve gene yalnızca temel halkaya dair bir şart, genel homoloji modüllerinin homolojik boyutlarının sonluluğunu vermektedir.

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