Commun. Fac. Sci. Univ. Ank. Serie A_1
V. 38, Number 1-2, pp. 53-58 (1989)

TWO COROLLARIES OF HOLDITCH'S THEOREM FOR ONE-PARAMETER CLOSED SPATIAL MOTIONS

R.GÜNEŞ*, S. KELEŞ**, H.H.HACISALİHOĞLU***
*, **: Science önd A U S Faculty, Inönü Univ.Malatya, Turkey.
***: Science Faculty, Ankara Univerlity, Ankara - Turkey.

ABSTRACT

Hacısalihoğlu [1] obtained two important corollaries by using Holditch's Theorem which is well-known for one-parameter closed planar motions. Müller [2] generalized Holditch's Theorem for space motions and the points in the space. In this paper, we extended the corollaries of Hacısalihoğlu by applying Müller's technique to space motions and the points in the space.

Introductıon

Let $\{0;\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{0'; \vec{e}'_1, \vec{e}'_2, \vec{e}'_3\}$ be two right-handed sets of the orthogonal unit vectors which are rigidly linked to moving space R and the fixed space R', recpectively, and denote the matrices E, E' by ;

$$E = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}, \quad E' = \begin{bmatrix} \vec{e}'_1 \\ \vec{e}'_2 \\ \vec{e}'_3 \end{bmatrix}.$$

We may then write

$$E = AE' \text{ or } E' = A^{-1} E = A^T E \qquad (1)$$

where $A = [a_{ij}]$ is a positive orthogonal 3x3 matrix, and the exponent-l indicates the inverse, also the superscript T the transpose. The elements a_{ij} of the matrix A will be regarded as functions of a real single parameter t and we will write $A=A(t)$ to restrict the discussion to one-parameter motions. If the matrix function A(t) is periodic, say $A(t+2\pi)= A(t)$, $\forall t$, the motions R/R' is closed, otherwise it is open.

ISSN 02571-081 A.Ü. Basımevi

During the closed motions R/R'; the orbits of the points of R and R' are closed curves. Since the matrix A is a positive orthogonal matrix we may write $AA^T = I$, where I is the unit matrix. This equation, by differentiation with respect to t, yields $dAA^T + AdA^T = 0$. This relation shows that the matrix $\Omega = dAA^T$ is antisymetric. Then we may write

$$\Omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

Differentiation of the first of equation of (1), with respect to t, yields

$$dE = \Omega E . \tag{2}$$

We may write

$$\vec{\omega} = \sum_{i=1}^{3} \omega_i \vec{e_i} \tag{3}$$

and

$$d\vec{e_i} = \vec{\omega} \Lambda \vec{e_i} \tag{2'}$$

since the matrix Ω is antisymetric. Where Λ denotes the vector product and $\vec{\omega}$ is called the instantaneous rotation vector of the motion R/R'. The vector which belongs to the translation part of the motion R/R' is

$$\vec{\bar{\omega}} = d\vec{u} = \bar{\omega}_1\vec{e}_1 + \bar{\omega}_2\vec{e}_2 + \bar{\omega}_3\vec{e}_3 ;$$
$$\bar{\omega}_i = du_i = \omega_j u_k - \omega_k u_j \quad . \tag{4}$$

Suppose that X is a fixed point in the moving space R. The position vectors of the point X are $\vec{x}$ and $\vec{x}'$, with respect to the moving space R and the fixed space R', respectively. Then, we may write

$$\vec{x}' = \vec{u} + \vec{x}$$

and

$$d\vec{x}' = d\vec{u} + d\vec{x}$$

or

$$d\vec{x}' = \vec{\bar{\omega}} + \vec{\omega} \Lambda \vec{x} = \sum_{i=1}^{3} \zeta_i \vec{e_i} \tag{5}$$

where $\zeta_i = \bar{\omega}_i + \omega_j x_k - \omega_k x_j$ [2].

Let (X) be the trajectory of a fixed point X of the moving space R in the fixed space R′ under the one-parameter closed motion R/R′.

The area vector of the curve (X) in the fixed space R′ is

$$\vec{V}_X = \oint \vec{x}' \wedge d\vec{x}' \tag{6}$$

[2], where the integrations are taken along the closed curve (X) on R′.

The area bounded by the orthogonal projection of the closed curve (X), in the direction of $\vec{e}'$ on a planea P is

$$F_X = F_0 + \sum_{i=1}^{3} B_i X_i + \left(\sum_{i=1}^{3} F_{ii} \right) \left(\sum_{i=1}^{3} x_i^2 \right) - \sum_{i,\, j=1}^{3} F_{ij} x_i x_j \tag{7}$$

where $\|\vec{e}'\| = 1$. If it is chosen a suitable co-ordinate systems then the equation (7) can be written as

$$F_X = F_0 + \sum_{i=1}^{3} A_i x_i^2 \tag{8}$$

where $A_i = F_{jj} + F_{kk}$, (i,j,k cyclic), [2].

Let X and Y be two different fixed points in the moving space R. Suppose that Z is a point with the components

$$z_i = \lambda x_i + \mu y_i, \ \lambda + \mu = 1 \tag{9}$$

on the straight line XY. The point Z has an orbit (Z) in R' during one-parameter colesed motion. The area bounded by the orthogonal projection of the closed curve (Z) on the plane P is, [2].

$$F_Z = \lambda F_X + \mu F_Y - \lambda\mu \sum_{i=1}^{3} A_i (x_i - y_i)^2 \tag{10}$$

The distance between the points X and Y can be given by the the metric

$$D^2(X,Y) = \varepsilon \sum_{i=1}^{3} A_i (x_i - y_i)^2 \tag{11}$$

such that $\varepsilon = \pm 1$. If

$$\lambda = \frac{D(Z,Y)}{D(X,Y)}, \ \mu = \frac{D(X,Z)}{D(X,Y)}. \tag{12}$$

Equation (10) can be written as

$$F_Z = \frac{1}{D(X,Y)} \varepsilon \{F_X D(Z,Y) + F_Y D(X,Z)\} - \varepsilon D(X,Z)\, D(Z,Y). \qquad (13)$$

Suppose that the fixed points X and Y in the moving space draw same colesed curve under the one-parameter colesed motion R/R'. In this case, $\vec{V}_X = \vec{V}_Y$ and that's why $F_X = F_Y$. Thus the equation (13) reduces to

$$F_X - F_Z = \varepsilon D(X,Z)\ D(Z,Y) \qquad (14)$$

which gives Holditch's Theorem.

Two Corollaries Of Holditch's Theorem For Oneparameter Closed Space Motions

Given an one-parameter closed spatial motion and a fixed straight line k in the moving space R. By choosing four arbitrary fixed points M,N,X and Y on the line k, let two of them move on the same curve (L), while the other two describe different curves (X) and (Y).

Corollary 1. Let Fand F′ be the areas between the projections of the curves (L) and (X) and of the curves (L) and (Y), respectively. Then the ratio F/F' depends only on the relative positions of these four points.

Proof: According to (14), the area F′ between the projection of the curves (L) and (Y) is

$$F' = F_M - F_Y = \varepsilon D(M,Y)\ D(Y,N) \qquad (15)$$

and the area F between the projection of the curves (L) and (X) is

$$F = F_M - F_X = \varepsilon D(M,X)\ D(X,N). \qquad (16)$$

Then, joining the last two equalities the ratio F/F' can be obtained as

$$\frac{F}{F'} = \frac{D(M,X)\ D(X,N)}{D(M,Y)\ D(Y,N)}$$

or

$$\frac{F}{F'} = \left(\frac{D(M,X)}{D(M,Y)}\right)^2 \frac{D(M,Y)\ D(X,N)}{D(M,X)\ D(Y,N)} \ . \qquad (17)$$

The invariant (17) does not depend on the curve (L) and length of MN. It depends only on the choice of the points X and Y on MN. Since $X \neq Y$, it follows that

$$\frac{D(M,Y)}{D(M,X)} \neq 1.$$

Denote

$$l = \frac{D(M,Y)}{D(M,X)} \cdot \frac{D(X,N)}{D(Y,N)} . \tag{18}$$

l is the cross ratio of the four points M, N, X, Y, i,e. $l = (MNXY)$.

This corollary is the re-stated form of the corollay which has been given for one-parameter closed planar motions. Thus the corollary in [1] is generalized to the points of space and spatial motions.

Corollary 2. Let M,N,A and B be four different fixed points in the moving space R. Suppose that the line segments MN and AB meet at the point X. Then the pairs of the points M,N and A,B are on the same curve or the areas bounded by the orthonogal projection of the closed orbits of the pairs on the Plane P are equal if and only if

$$D(M,X)\ D(X,N) = D(A,X)\ D(X,B).$$

Proof: Given the different fixed points M,N,A and B. Hence the segments MN,AB,MA,NA,MB and NB are constant. Suppose the segments MN and AB meet at the fixed point X. While the pairs of the points A,B and M,N draw a closed curve in the fixed space R′ under the motion R/R′ the point X also draws a different closed curve. The projected curves, in the direction of the vecor $\vec{e}'$ on the plane P of these closed curves are closed as well.

By using equation (14), the ring area between these projected curves may be obtained as

$$F_M - F_X = \varepsilon D(M,X)\ D(X,N) \tag{19}$$

and similarly

$$F_A - F_X = \varepsilon D(A,X)\ D(X,B). \tag{19′}$$

Using the equations (19) and (19′) one can write

$$F_M - F_A = \varepsilon \{D(M,X)\ D(X,N) - D(A,X)\ D(X,B)\}. \tag{20}$$

Since $F_M = F_A$, from the equation (20), we have

$$D(M,X)\ D(X,N) = D(A,X)\ D(X,B).$$

This proposition implies that the points M, N, A, B lie on the same circle of the moving space R. This is a particular case of the steiner Theorem.

Conversely, suppose that

$$D(M,X)\ D(X,N) = D(A,X)\ D(X,B).$$

Thus it can be written

$$\varepsilon D(M,X)\ D(X,N) = \varepsilon D(A,X)\ D(X,B)$$

where $\varepsilon = \pm 1$. From (14), we have

$$F_M - F_X = F_A - F_X$$

or

$$F_M = F_A.$$

This implies that the pairs of the points M,N and A,B are on the same curve or the areas bounded by the orthogonal projected curves of the closed orbits of these pairs on a plane are equal.

REFERENCES

[1] HACISALİHOĞLU. H.H. On Closed Spherical Motions. Quarterly of Applied Math., 1971, pp: 269–275.

[2] H.R.MÜLLER. Erweiterung des Satzes von Holditch für geschlossene Raumkurven. Anbhandl. Braunschw. Wiss Ges. 31 (1980), 129–135.

[3] H.R.MÜLLER. Kinematik Dersleri. A.Ü. Fen Fakültesi yayınları Um.96-Mat.27. Ankara. 1963.