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# TWO COROLLARIES OF HOLDITCH'S THEOREM FOR THE PARTS OF SURFACE IN $E^3$

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ABSTRACT

Hacısalihoğlu [1] obtained two important corollaries by using Holditch's Theorem which is well-known for one-parameter closed planar motions. Müller [2] re-stated Holditch's Theorem for the parts of surface in $E^3$. In this paper, we extended the corollaries of Hacısalihoglu by applying Müller's technique to spatial motions and the parts of surface in $E^3$.

## I. Introduction

Let us assume that $\psi$ is a surface and $X = (u,v)$ is a point on $\psi$ in the 3-dimensional Euclidean space $E^3$. The position vector X can be written

$$\vec{OX} = \vec{x} = \sum_{i=1}^{3} x_i \vec{e}_i \tag{1}$$

where $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an *orthonormal* frame in $E^3$. Let $\Gamma$ be a connected domain on $\psi$ and the functions $X_i(u.v)$ on this domain be differentiable. If we denoted by

$g_{11} = \langle \vec{x}_u, \vec{x}_u \rangle$, $g_{12} = \langle \vec{x}_n, \vec{x}_v \rangle$, $g_{22} = \langle \vec{x}_v, \vec{x}_v \rangle$, $w^2 = g_{11}g_{22} - g_{12}^2$
then we can have

$$dF = du \Lambda dv \tag{2}$$

and dF is called the *scalar area element* of $\psi$ on $\Gamma$, [2]. The unit normal vector field of the surface $\psi$, at the point X, is

$$\vec{n} = \frac{1}{w} (\vec{x}_u x \vec{x}_v). \tag{3}$$

where, $\langle , \rangle$ denotes the scalar product, $\Lambda$ denotes the alternate product and x denotes the cross product.

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Thus the vectoral area element can be defined as:

$$\vec{dF} = \frac{1}{2} \vec{dx} \hat{x} \vec{dx} = \vec{n} dF \tag{4}$$

where $\hat{x}$ denotes the cross product between vectors and also the alternate product between 1– forms [3]. The vector $\vec{F}$ which defined by

$$\vec{F} = \iint_{\Gamma} \vec{dF} \tag{5}$$

is called the area vector of $\psi$ on the domain $\Gamma$. Let P be a plane passing through the origin point. The orthogonal projection of vector $\vec{F}$ in the direction of $\vec{e} = \vec{e}_3$ on the plane P is

$$<\vec{e}, \vec{F}> = <\vec{e}, \iint_{\Gamma} \vec{dF}> = F^n. \tag{6}$$

Hence, $F^n$ is called the projection area of $\psi$.

## II. The Holditch's Theorem for The Parts of Surface in $E^3$

Let $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{O'; \vec{e'}_1, \vec{e'}_2, \vec{e'}_3\}$ be two right-handed sets of orthonormal unit vectors which are rigidly linked to moving space R and the fixed space R', respectively. It is shown that ([3])

$$d\vec{e}_i = w_j \vec{e}_k - w_k \vec{e}_j \text{ , (i, j, k cyclic).} \tag{7}$$

If the Pfaff forms $w_i$, $1 \leq i \leq 3$, are the functions of the real variables u and v, then (7) defines a 2–parameter rotation motions. These Pfaff forms are not independent of each other. Therefore, they must satisfy the conditions of integrability. Hence,

$$dw_i = w_j \Lambda w_k. \tag{8}$$

Let X be a fixed point on the moving space R, then

$$\overrightarrow{O'X} = \overrightarrow{O'O} + \overrightarrow{OX}$$

or

$$\vec{x}' = \vec{u} + \vec{x}. \tag{9}$$

Therefore, we obtain

$$d\vec{x}' = d\vec{u} + d\vec{x} = \vec{\sigma} + \vec{w}\Lambda\vec{x} \tag{10}$$

where $d\vec{u} = \vec{\sigma}$ and $d\vec{x} = \vec{w}\vec{x}$. If the components of $d\vec{x}'$, are denoted by $\tau_i$, $1 \leq i \leq 3$, then

$$dx'_i = \tau_i = \sigma_i + x_j w_k - x_k w_j. \tag{11}$$

The volume element of the point X is

$$dJ_X = \tau_1 \Lambda \tau_2 \Lambda \tau_3 \tag{12}$$

[1]. The equation (12) is a linear quadratic from with respect to $x_i$, $1 \leq i \leq 3$. If some suitable coordinates are choosen, then

$$\sigma_k \Lambda \sigma_i \Lambda w_i + \sigma_k \Lambda \sigma_j \Lambda w_i = 0$$

and

$$\sigma_i \Lambda w_i \Lambda w_k - \sigma_j \Lambda w_j \Lambda w_k = 0. \tag{13}$$

Let $\alpha_i$ be given by

$$\alpha_i = \sigma_i \Lambda w_j \Lambda w_k, \; 1 \leq i \leq 3, \tag{14}$$

from equation (13) the volume element of the point X becomes

$$dJ_X = dJ_0 + \sum_{i=1}^{3} \alpha_i x_i^2 \tag{15}$$

where, $dJ_0 = \sigma_1 \Lambda \sigma_2 \Lambda \sigma_3$ is the volume element of the point O during the motion. If we write

$$\sigma_i = s_{i1} w_1 + s_{i2} w_2 + s_{i3} w_3 \tag{16}$$

then (13) nda (14) give us

$$s_{ij} + s_{ji} = 0, \; s_{ii} s_{jk} = 0.$$

If $dJ_0 \neq 0$, then $s_{ii} \neq 0$, and $s_{ij} = 0$. Further, if we denote that $s_{ii} = s_i$, we obtain,

$$\sigma_i = s_i w_i, \; \alpha_i = s_i w_1 \Lambda w_2 \Lambda w_2 \tag{17}$$

Thus

$$dJ_X = (s_1 s_2 s_2 + \sum_{j=1}^{3} s_j x_i^2)\ w_1 \Lambda w_2 \Lambda w_3 \tag{18}$$

[2]. Assume that a fixed point $X \in R$ draws a closed surface $\psi_x$ during the motion. The vectoral area element, on connected subdomain $\Gamma \subset \psi_X$ is

$$dF'_X = \frac{1}{2} (d\vec{x}' \hat{x} d\vec{x}') = \vec{e}_1 \tau_2 \Lambda \tau_3 + \vec{e}_2 \tau_3 \Lambda \tau_1 + \vec{e}_3 \tau_1 \Lambda \tau_2. \tag{19}$$

Hence, the area vector can be given as [2]

$$F'_X = \int_\Gamma \int d\vec{F}'_X = \sum_{i=1}^{3} \vec{e}_i \int_{\rightarrow \Gamma} \int \tau_j \Lambda \tau_k = \sum_{i=1}^{3} \vec{e}_i f_i. \tag{20}$$

After some calculations we have

$$\tau_i \Lambda \tau_j = (s_i s_j + x^2_k)\ w_i \Lambda w_j + (x_k x_i - s_j x_j)\ w_j \Lambda w_k + (s_i x_i + x_j x_k) w_k \Lambda w_i$$

and

$$\iint_\Gamma \tau_i \Lambda \tau_j = \iint_\Gamma s_i s_j w_i \Lambda w_j + x^2_k \iint_\Gamma w_i \Lambda w_j + x_i x_k \iint_\Gamma w_j \Lambda w_k - x_j \iint_\Gamma s_j w_j \Lambda w_k$$
$$+ x_i \iint_\Gamma s_i w_k \Lambda w_j + x_j x_k \iint_\Gamma w_k \Lambda w_i.$$

Lef denote that

$$2a_{ij} = \int \int w_i \Lambda w_j,\ 2b_{ij} = \int \int s_i w_i \Lambda w_j,\ 2c_{ij} = \int \int s_j w_i \Lambda w_j,$$

$$h_{ij} = \int \int s_i s_j w_i \Lambda w_j,\ \vec{a} = \vec{e}_1 a_{22} + \vec{e}_2 a_{21} + \vec{e}_3 a_{12}. \tag{21}$$

Therefore the components $f_i$ of the area vector $\vec{F}'_X$ are

$$f_i = f_i(X,X) = h_{jk} + 2c_{ij} x_j - 2b_{ki} x_k + 2x_i \langle \vec{a}, x \rangle. \tag{22}$$

Let X and Y be two fixed points of the moving space R and Z be a point on the line-segment $\overline{xy}$ such that the components of Z are

$$z_i = \lambda x_i + \mu y_i\ ,\ \lambda + \mu = 1,\ 1 \leq i \leq 3. \tag{23}$$

Then the components of the area vector of the surface drawn by the point Z, are

$$f_i(Z,Z) = h_{jk} + 2c_{ij} z_i - 2b_{ki} z_k + 2z_i \langle \vec{a}, \vec{z} \rangle. \tag{24}$$

From (23) we have

$$f_i(Z,Z) = \lambda^2 f_i(X,X) + 2\lambda\mu f_i(X,Y) + \mu^2 f_i(Y,Y).$$

Here the vector $\vec{F}'_{XY} = \sum_{i=1}^{3} \vec{e}_i f_i(X,Y)$ is also mixed area vector of the surface [2]. Since

$$\vec{F}'_X - 2\vec{F}_{XY} + \vec{F}_Y = 2<\vec{a},\vec{x}-\vec{y}> (\vec{x}-\vec{y}) = \vec{M} \tag{25}$$

we can write

$$\vec{F}'_Z = \lambda\vec{F}_X + \mu\vec{F}_Y - \lambda\mu\vec{M}. \tag{26}$$

Now, assume that the end points of the line-segment $\overline{XY}$ draw the same surface during the motion. So $\vec{F}_X = \vec{F}_Y$. from (26) this implies that

$$\vec{F}'_Z = \vec{F}'_X - \lambda\mu\vec{M}. \tag{27}$$

The projected area of bounded domain in the direction of $\vec{e} = \vec{e}_3$ on to plane P is

$$F^n_Z = <\vec{e}, \vec{F}'_Z> = <\vec{e}, \vec{F}'_X> - \lambda\mu <\vec{e},\vec{M}>. \tag{28}$$

Let's define the distance between X and Y as

$$D^2(X,Y) = \varepsilon <\vec{e}, \vec{M}>, \varepsilon=\pm 1, \tag{29}$$

for the points X and Y of R [2]. Since we have

$$\lambda = \frac{D(Y,Z)}{D(X,Y)}, \mu = \frac{D(X,Z)}{D(X,Y)}$$

the equation (28) reduces to

$$<\vec{e}, \vec{F}'_X> - <\vec{e}, \vec{F}'_Z> = \varepsilon D(X,Z) D(Z,Y)$$

or

$$F^n_X - F^n_Z = \varepsilon D(X,Z) D(Z,Y). \tag{30}$$

This equation gives us the Theorem of Holditch.

## III. Two Corollaries of Holditch's Thorem for The Parts of a Surface In $E^3$.

Now, in the moving space R, let us take arbitrary points M, N, X and Y, where all of them on a fixed straight line k. Let the points X and Y be on the segment line $\overline{MN}$. While the points M,N draw a same surface part $\Phi$ during the motion, the points X and Y also draw two different surface parts $\psi_X$ and $\psi_Y$, respectively. Let $F^n_1$ be the area between the orthogonal projection of $\Phi$ and $\psi_X$ and $F_2^n$ be the area between projection orthogonal projeetion of $\Phi$ and $\psi_Y$.

Hence we can give the following corollaries:

*Corollary* 1. The ratio $F_1^n/F_2^n$ depends only on relative positions of these four points to each other.

Proof: From (30) we have

$$F_1^n = F^n_M - F^n_X = \varepsilon\, D(M,X)\, D(X,N) \tag{31}$$

and

$$F_2^n = F^n_M - F^n_Y = \varepsilon\, D(M,Y)\, D(Y,N). \tag{31'}$$

So we have the followinq ratio.

$$\frac{F_1^n}{F_2^n} = \frac{D(M,X)}{D(M,Y)} \quad \frac{D(X,N)}{D(Y,N)}$$

or

$$\frac{F_1^n}{F_2^n} = \left|\frac{D(M,X)}{D(M,Y)}\right| \frac{D(M,Y)}{D(M,X)} \quad \frac{D(X,N)}{D(Y,N)}. \tag{32}$$

This ratio does not depend on the surface part $\Phi$. It only depends on the choice of the points X and Y on the line-segment $\overline{MN}$. Since $X \neq Y$ we have

$$\frac{D(M,Y)}{D(M,X)} \neq 1.$$

If we denote $l = \dfrac{D(M,Y)}{D(M,X)} \quad \dfrac{D(X,N)}{D(Y,N)}$

$l$ is the cross ratio of the four points M,N,X,Y, i.e. $l = (MNXY)$.

*Corollary* 2. Let M,N,A and B, be four different fixed points in the moving space R. Suppose that the line segments $\overline{MN}$ and $\overline{AB}$ be meet at a point X. Then, each of pairs of the points M,N, and A,B are on the same closed surface parts or on the different surface parts but the areas of the arthogonal projection of these closed sufface parts are equal if and only if

$$D(M,X)\ D(X,N) = D(A,X)\ D(X,B). \qquad (33)$$

*Proof:* Since the points M,N,A and B are fixed on the moving space R the line-segments $\overline{MN}$, $\overline{AB}$, $\overline{MA}$, $\overline{NA}$, $\overline{MB}$ and $\overline{NB}$ are alsa fixed in R. Suppose the line segments $\overline{MN}$ and $\overline{AB}$ be meet at a pointe X. Now, let's move these line segments such that the end points of them draw the same surface part in the fixed space R′. During the motion, the point X also draws another closed surface part The difference of the orthogonal projection area of these surface parts on the plane P are from (30):

$$F^n{}_M - F^n{}_X = \varepsilon\ D(M,X)\ D(X,N)$$

and

$$F^n{}_A - F^n{}_X = \varepsilon\ D(A,X)\ D(X,B).$$

Thus we obtain

$$F^n{}_M - F^n{}_A = \varepsilon\ \{D(M,X)\ D(X,N) - D(A,X)\ D(X,B)\}. \qquad (34)$$

Since $F^n{}_M = F^n{}_A$, the equation (34) gives us

$$D(M,X)\ D(X,N) = D(A,X)\ D(X,N).$$

Conversly, suppose that

$$D(M,X)\ D(X,N) = D(A,X)\ D(X,B).$$

On the other hand we have

$$\varepsilon\ D(M,X)\ D(X,N) = \varepsilon\ D(A,X)\ D(X,B)$$

where $\varepsilon = \pm 1$. From (30), we can write

$$F^n{}_M - F^n{}_X = F^n{}_A - F^n{}_X$$

or $F^n{}_M = F^n{}_A$

This implies that the pairs of the points M,N and A,B are on the same surface part or on the different surface parts but whose orthogonal projected areas on the plane are eqnal.

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