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GENERALIZED CROSS PRODUCT in R^6 and R^m, $m = \frac{n(n-1)}{2}$

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ABSTRACT

In this study in the space R^6, the cross–product was defined as analogous vector–product in R^3. We showed that this product makes R^3 a Lie algebra. Therefore, it was showed that the Lie algebras $(R^6,0)$ and $(A_4, [,])$ are isomorphic. As a generalization, in the space of dimension $m = n(n-1)/2$, cross–product can be given as

$$R^m \times R^m \rightarrow R^m, \; xoy = J^{-1} [J(X), J(Y)]$$

where $J = R^m \rightarrow An$ is Lie algebra isomorphism. At the end, we showed that the cross – product we defined is vector product well known for $n = 3$.

INTRODUCTION

Studying kinematics, the set $A_3 = \{A \in M\,(3x3, R)/A^{-1} = A^T\}$ is very important. A_3 is the Lie algebra with the product $[A, B] =$ AB–BA. And $(A_3, [\,,]$ is the Lie algebra of orthogonal matrices of order 3 x 3, 0(3). Moreever (R^3, x) is the other Lie algebra and this algebra is isomorphic to $(A_3, [\,,])$ [1]. This isomorphism gave to use an inspiration. Is it possible to define a product in A_m, $m = \frac{n\,(n-1)}{2}$, which makes R^m a Lie algebra under a isomorphism? We showed that it is possible. At the first we studied in R^6 (which is isomorphic to A_4).

The product we will define is handy to study the theory of dual numbers and of the dual sphere [2]. For this purpose we used the properties of anti-symmetric mappings, permutations and determinant function [3].

2. Let S_6 be permutation group of the set $M = \{1,2,\ldots.6\}$. We define a relation on S_6 as following. For every σ, λ are elements of S_6,

$$\sigma \backsim \lambda \rightarrow \sigma(5) = \lambda(5) \text{ and } \sigma(6) = \lambda(6)$$

This relations is a equivalence relation on S_6. So we have equivalence class. Each equivalence class is known by an element of the set

$$PC^6{}_2 = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$$

For $(\lambda, \beta) \in PC^6{}_2$, the equivalence class is shown by $\sigma_{\lambda\beta}$ Half of the permutations of $\sigma_{\lambda\beta}$ are even the others are odd. So $\sigma_{\lambda\beta} = (\sigma_{\lambda\beta})_e \cup (\sigma_{\lambda\beta})_0$. But we will use only even permutations and write $\sigma_{\lambda\beta}$ for $(\sigma_{\beta\lambda})_e$.

3. Cross product in R^6

Let $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the standard base of R^6. We define the product $\otimes$ as the following.

$\otimes$	e_1	e_2	e_3	e_4	e_5	e_6
e_1	0	e_4	$-e_5$	e_2	e_3	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3
e_3	e_5	e_6	0	0	$-e_1$	$-e_2$
e_4	$-e_2$	e_1	0	0	$-e_6$	$-e_5$
e_5	$-e_3$	0	e_1	e_6	0	$-e_4$
e_6	0	$-e_3$	e_2	$-e_5$	e_4	0

Tableau 1.

If $\sigma \in \sigma_{(\lambda,\beta)}$, then $e_{\sigma(5)} \otimes e_{\sigma(6)} = e_\lambda \otimes e_\beta$. Also we have

$$\sum_{\sigma \in \sigma_{\lambda\beta}} \det [X, Y, e_{\sigma(1)}, \ldots\ldots, e_{\sigma(4)}]\, e_{\sigma(5)} \otimes e_{\sigma(6)} =$$

$$= 12 \det [X, Y, e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_{\sigma(4)}]\, e_\lambda \, O \, e_\beta$$

where σ is an element of $\sigma_{(\lambda,\beta)}$. In such a manner that, det $[X, Y, e_{\sigma(1)}, \ldots, {}_{\sigma(4)}] =$

$$= \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ \delta_{\sigma(1)1} & \delta_{\sigma(2)1} & \delta_{\sigma(3)1} & \delta_{\sigma(4)1} & \delta_{\sigma(5)1} & \delta_{\sigma(6)1} \\ \delta_{\sigma(1)2} & \delta_{\sigma(2)2} & \delta_{\sigma(3)2} & \delta_{\sigma(4)2} & \delta_{\sigma(5)2} & \delta_{\sigma(6)2} \\ \delta_{\sigma(1)3} & \delta_{\sigma(2)3} & \delta_{\sigma(3)3} & \delta_{\sigma(4)3} & \delta_{\sigma(5)3} & \delta_{\sigma(6)3} \\ \delta_{\sigma(1)4} & \delta_{\sigma(2)4} & \delta_{\sigma(3)4} & \delta_{\sigma(4)4} & \delta_{\sigma(5)4} & \delta_{\sigma(6)4} \end{vmatrix}$$

$$= \det \begin{bmatrix} x_{\sigma(1)} & x_{\sigma(2)} & x_{\sigma(3)} & x_{\sigma(4)} & x_{\sigma(5)} & x_{\sigma(6)} \\ y_{\sigma(1)} & y_{\sigma(2)} & y_{\sigma(3)} & y_{\sigma(4)} & y_{\sigma(5)} & y_{\sigma(6)} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= x_{\sigma(5)} \; y_{\sigma(6)} - x_{\sigma(6)} \; y_{\sigma(5)}.$$

So we have

1. Definition: For every $X.Y \in R^6$ the product

$$XOY = \sum_{\sigma(S_6/\backsim)_e} \det\,[X,Y,\, e_{\sigma(1)},\, e_{\sigma(2)},\, e_{\sigma(3)},\, e_{\sigma(4)}]\; e_{\sigma(5)} \otimes e_{\sigma(6)}$$

is called cross-product in R^6.

2. Proposition: If $X, Y \in R^6$ then

1) $X \otimes Y = -Y \otimes X$, $X \otimes X = 0$ (anti-symetry-property)

2) The product $\otimes$ is bilinear.

Proof: It is easy to show by using the properties of determinant function.

3. Proposition: If $X, Y \in R^6$, $\sigma \in \sigma_{\lambda\beta}$ and $(\lambda, \beta) \in PC^6{}_2$ then

$$X \otimes Y = \sum_{\sigma} (X_{\sigma(5)} \; Y_{\sigma(6)} - X_{\sigma(6)} \; Y_{\sigma(5)}) \; e_{\sigma(5)} \otimes e_{\sigma(6)}$$

With the direct calculation, $X \otimes Y$ can be written in compenent of X and Y as follows

$$X \otimes Y = (x_4y_2 - y_4x_2 + x_5y_3 - y_5x_3)\vec{e}_1 + (x_1y_4 - y_1x_4 + x_6y_3 - y_5x_3)\vec{e}_2$$
$$+ (x_1y_5 - y_1x_5 + x_2y_6 - y_2x_6)\vec{e}_3 + (x_2y_1 - y_2x_1 + x_6y_5 - y_6x_5)\vec{e}_4$$
$$+ (x_4y_6 - y_4x_6 + x_3y_1 - y_3x_1)\vec{e}_5 + (x_3y_2 - y_3x_2 + x_5y_4 - y_5x_4)\vec{e}_6$$

4. Theorem: The product $\otimes$ in R^6 has the following properties.

1) For every $X, Y \in R^6$,

$$< X \otimes Y, X > = 0, < X \otimes Y, Y > = 0$$

2) For every X, Y, Z $\in R^6$

$$< X \otimes Y, Z > = < X, Y \otimes Z >$$

3) For every X, Y, Z $\in R^6$

$$X \otimes (Y \otimes Z) + Y \otimes (Z \otimes X) + Z \otimes (X \otimes Y) = 0$$

Proposition 2 and theorem 3 show that $(R^6, \otimes)$ is a Lie algebra.

5. Isomorphism between the Lie algebras $(R^6, \otimes)$ and $(A_4, [,])$

Let A_4 be the set of all anti-symmetric matris order of 4x4. The system $(A_4, \oplus, (R, +, .), \odot)$ is a vector space of dimensions 6 and A_4 is a Lie algebra with the Lie bracket operator $[,]$,

$$[,] = A_4 x A_4 \to A_4, \; [X, Y] = X.Y - Y.X$$

Therefore we have the Lie algebra $(R^6, \otimes)$. We can define a mapping J between R^6 and A_4 as follows.

$$J = R^6 \to A_4$$

$$J(x_1,x_2,x_3,x_4,x_5,x_6) = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{bmatrix}$$

5. Theorem: The mapping J is a Lie algebra isomorphism.

Proff: The mapping J is one-to-one and onto. Moreover, for

$$X = \sum_{i=1}^{6} x_i \,.\, \vec{e}_i \;, \; Y = \sum_{i=1}^{6} y_i \,.\, \vec{e}_i \,,$$

$$J(X \otimes Y) = J\, x_4y_2 - y_4x_2 + x_5y_3 - y_5x_3, x_1y_4 - y_1x_4 + x_6y_3 - y_6x_3, \\ x_1y_5 - y_1x_5 + x_2y_6 - y_2x_6, x_2y_1 - y_2x_1 + x_6y_5 - y_6x_5 \\ x_4y_6 - y_4x_6 + x_3y_1 - y_3x_1, x_3y_2 - y_3x_2 + x_5y_4 - y_5x_4)$$

$$= \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{bmatrix} \begin{bmatrix} 0 & y_1 & y_2 & y_3 \\ -y_1 & 0 & y_4 & y_5 \\ -y_2 & -y_4 & 0 & y_6 \\ -y_3 & -y_5 & -y_6 & 0 \end{bmatrix} -$$

$$\begin{bmatrix} 0 & y_1 & y_2 & y_3 \\ -y_1 & 0 & y_4 & y_5 \\ -y_2 & -y_4 & 0 & y_6 \\ -y_3 & -y_5 & -y_6 & 0 \end{bmatrix} \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{bmatrix} = XY - YX$$

The isomorphism J between (R^6, $\otimes$) and (A_4, [,]) allows to present an anologous product on R^m, where m is the boy A_n. As we know the dimension of A_n is $\frac{n(n-1)}{2}$. So we have the product $\otimes_m$ as

$$\otimes_m = \begin{array}{ccc} R^m \times R^m & \xrightarrow{O_m} & R^m \\ J \downarrow \quad \downarrow J & & \uparrow J^{-1} \\ A_n \times A_n & \longrightarrow & A_n \end{array}$$

$$X \otimes Y = J^{-1} (J(x), J(y))$$

$\otimes_m$, has the properties of vector-product. That is

1) $\otimes_m$ is anti-symmetric

2) For every $X, Y \in R^m$, $<X \otimes Y, X> = 0$ and $<X \otimes_m Y, Y> = 0$

3) The product $\otimes_m$ is bi-linear.

4. The special case for n = 3.

Now we will show that the product we defined in definition 1 is the vector product in R^3 well known. Consider the set M = {1, 2, 3}. Let S_e (3) be the all even permutations of M, i.e.

$$S_e(3) = \{(1,2,3,)\ (2,3,1),\ (3,1,2)\}.$$

Moreeever $S_e(3)$ is the set all of permutations which has the properties $\sigma(2) = \tau(2)$, $\sigma(3) = \tau(3)$ for every $\sigma, \tau \in S_e(3)$. So, For $X = (x_1, x_2 x_3)$, $Y = (y_1, y_2, y_3) \in R^3$ and $\sigma \in S_e(3)$ we have

$$X \otimes Y = \sum_{\sigma} \det (X, Y, e_{\sigma(1)})\ e_{\sigma(2)} \otimes e_{\sigma(3)} \quad \ldots..$$

If we set $e_1 = e_1$, $e_2 = -e_3$, $e_3 = -e_5$ from tableau. 1. then we can give the vector product as in tableau 2.

	e_1	e_2	e_3
e_1	0	e_3	$-e_2$
e_1	$-e_3$	0	e_1
e_1	e_2	$-e_1$	0

Tableau 2.

Clearly, we have

$$X \otimes Y = \det(X,Y,e_1)\vec{e}_1 + \det(X,Y,e_2)\vec{e}_2 + \det(X,Y,e_3)\vec{e}_3$$

$$= \sum_{i=1}^{1} \det(X,Y,\vec{e}_i)\, \vec{e}_i$$

ÖZET

Bu çalışmada, R^6 uzayında, R^3 teki vektörel çarpımın bir benzeri tanımlandı. Tanımlanan bu vektörel çarpımın R^6 uzayının bir Lie cebiri yaptığını gösterdik. Ayrıca $(R^6, \otimes)$ Lie cebiri ile $(A_4, [,])$ Lie cebirinin izomorfik oldukları ispatlandı. Bir genelleme olarak, $m = \frac{n(n-1)}{2}$ boyutlu uzaylarda

$$[,] : A_n \times A_n \rightarrow A_n$$

çarpımı yardımıyla, R^m uzayında bir genel vektörel çarpımın,

$$R^m \times R^m \rightarrow R^m,\ X \otimes Y = J^{-1}\,[J(X), J(Y)]$$

ile verilebileceği gösterildi.

REFERENCES

1. KARGER, ADOLF And NOVAK, JOSEF. Space Kinematics and Lie Groups. Gordon and Science Publishers S.A. 1985, New York.

2. HACISALİHOĞLU, H. HİLMİ. Hareket Geometrisi ve Kuaterniyonlar Teorisi. Gazi Üni. Fen Edebiyat Fakültesi Yayınları Mat. No: 2 1983–Ankara–Türkiye

3. GREUB, WERNER H. Linear Algebra. Springer –Verlag, 1975. Berlin.