Commun. Fac. Sci. Univ. Ank. Series A_1
V. 39. pp. 21-35 (1990)

ON THE LINEAR VECTOR FIELDS IN E^{2n+1}

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ÖZET

Karger ve Novak [1], E^3 de bir X lineer vektör alanının matrisi

$$\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$$

olduğuna göre X in integral eğrilerinin (i). rank [AC] = 3 olması halinde ortak eksenli ve aynı parametreli helis eğrileri, (ii). rank [AC] = 2 olması halinde birbirine paralel düzlemlerde yatan ve merkezleri bu düzlemlere dik bir eksen üzerinde bulunan çemberler, (iii) rank [AC] = 1 olması halinde paralel doğrular olduğunu göstermişlerdir.

Bu çalışmada Karger ve Novak'ın sonuçları $2n+1 > 3$ olmak üzere E^{2n+1} de genelleştirilmiştir. Sonuçların tamamen geçerli olduğu gösterilmiş ve bazı irdelemeler verilmiştir.

ABSTRACT

Karger and Novak [1] has shown that the integral curves of a linear vector field X on E^3 which has a matrix

are: $\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$

(i). Helices with common axes and the same parameter if rank [AC] = 3,

(ii). Circles which lie on planes parallel to each other, and which have centers on the axis perpendicular to those parallel planes, if rank [AC] = 2,

(iii) .Parallel straight lines, if rank [AC] = 1.

In this study, the results of Karger and Novak are extended to E^{2n+1} where $n>1$. It is shown that of all the results are also valid in this general case and some further elaborations are included.

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I. INTRODUCTIONS

Integral curves of a linear vector field of E^3 are depend on the rank of linear vector fields. The integral curves are circles or helices in the cases of the matrix of the linear vector ifeld has rank even or odd.

Recent years the theory of helices in higher dimensions has been completed so it is reasonable to study the theory of integral curves of a linear vector field.

We show that the theory of integral curves of a linear vector field in (2n+1)-dimensional Euclidean Space, $n > 1$, is the same of the case of $n=1$).

II. BASIC CONCEPTS

Let

$$\alpha: I \longrightarrow E^m$$
$$t \qquad\qquad \alpha(t)$$

be a parametric curve and X be a vector field in E^m.

If

$$\frac{d\alpha}{dt} = X(\alpha(t)) \ , \ \forall\ t \in I$$

is satisfied, then the curve α is called an integral curve of the vector field X.

Let V be a vector space over IR of dimension m. A vector field X on V is called linear if

$$X(\upsilon) = A(\upsilon) \ , \ \forall\ \upsilon \in V$$

where A is a linear mapping from V into V.

Let $A \in IR^{2n+1}_{2n+1}$ be a skew-symmetric matrix. Then we can choose

an orthonormal base ψ in IR^{2n+1} such that the matrix A reduces to the form as

$$\begin{bmatrix} 0 & \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ -\lambda_1 & 0 & \lambda_2 & \cdots & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_{2n-1} & 0 \\ 0 & 0 & 0 & \cdots & -\lambda_{2n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathrm{IR}^{2n+1}_{2n+1}$$

where $\lambda \in \mathrm{IR} - \{0\}$. If $C \in \mathrm{IR}_1{}^{2n+1}$, is a column matrix such as

$$C = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n+1} \end{bmatrix}$$

then we showed that the value of X at any point P of E^{2n+1} can be written as

$$\begin{bmatrix} X(P) \\ \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ & \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ \\ 1 \end{bmatrix},$$

where the matrix

$$\begin{bmatrix} A & C \\ & \\ 0 & 1 \end{bmatrix}$$

is called the matrix of the linear vector field X.

III. INTEGRAL CURVES OF A LINEAR VECTOR FIELD

III. 1. Linear Vector Fields in E^3.

Let X be a linear vector field in E^3 and an orthonormal frame of E^3 be $\{O; u_1, u_2, u_3\}$. Then the matrix in this frame can be written as

$$\begin{bmatrix} A & C \\ & \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 & a \\ -\lambda & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ rank } [AC] = 3.$$

Then the value of X at

a point P = (x, y, z) of E^3 is

$$\begin{bmatrix} X(P) \\ \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 & a \\ -\lambda & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

or

$$X(P) = (\lambda y + a, -\lambda x + b, c).$$

On the otherhand if a curve

$$\alpha : I \longrightarrow E^3$$
$$t \qquad \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

is an integral curve of X then we can write the differential equation

$$\frac{d\alpha}{dt} = X(\alpha(t)), \ \forall t \in I \tag{III.1.1}$$

or the system of differential equations

$$\left.\begin{aligned} \frac{dx}{dt} &= \lambda y + a \\ \frac{dy}{dt} &= -\lambda x + b \\ \frac{dz}{dt} &= c \end{aligned}\right\} . \tag{III.1.2}$$

For the shake of the shortnes, we can have $\lambda = 1$ and the system (III. 1.2) reduces to

$$\left.\begin{aligned} \frac{dx}{dt} &= y + a \\ \frac{dy}{dt} &= -x + b \\ \frac{dz}{dt} &= c \end{aligned}\right\} . \tag{III.1.3}$$

Then the solution of the last equation of this system is

$$z = ct + d. \tag{III.1.4}$$

For the solutions of the first two equations we derivate the second equation and obtain that

$$\frac{d^2y}{dt^2} = - \frac{dx}{dt}$$

and using the first equation we have

$$\frac{d^2y}{dt^2} + y = - a$$

which is a first order linear differential equation with constant coefficient.

We know that the solution of this equation is

$$y = A \cos t + B \sin t - a. \qquad \text{(III.1.5)}$$

On the otherhand the derivation of (III.1.5) and the second equation of (III.1.3) give us that

$$x = A \sin t - B \cos t + b. \qquad \text{(III.1.6)}$$

Thus the integral curves of X can be written as

$$\alpha(t) = (A \sin t - B \cos t + b, A \cos t + B \sin t - a, ct + d). \qquad \text{(III.1.7)}$$

This is a familly of inclined curves with common axes and the same parameter since we have

$$H = \frac{k_1}{k_2} = \frac{1}{c} \sqrt{A^2 + B^2} ,$$

where k_1 and k_2 are the curvatures of the curve and H is constand for each one of the curves.

Now assume the case that rank $[AC] = 2$.

In this case $c = 0$ since we know that $\lambda \neq 0$. So (III.1.7) gives us the equation of integral curves as

$$\alpha(t) = (A \sin t - B \cos t + b, A \cos t + B \sin t - a, d). \qquad \text{(III.1.7)}'$$

The curves are the circles each one of which lies on the parallel planes and the centres of these circles are located on an axis perpendicular to those parallel planes.

Finally, assume that rank $[AC] = 1$.

In this case we have that $\lambda = 0$ and the system (III.1.2) reduces to the system

$$\left.\begin{array}{l} \dfrac{dx}{dt} = a \\[2ex] \dfrac{dy}{dt} = b \\[2ex] \dfrac{dz}{dt} = c \end{array}\right\} . \qquad (III.1.2)'$$

Then the solution of this system is

$$\alpha(t) = (at + d_1,\ bt + d_2,\ ct + d_3). \qquad (III.1.7)''$$

These integral curves are the parallel straight lines.

III.2. Linear Vector Fields In E^{2n+1}

III.2.1. The General Case

Let $A \in IR^{2n+1}_{2n+1}$ be a skew-symmetric matrix and $C \in IR_1^{2n+1}$ be a column matrix such that

$$A = \begin{bmatrix} 0 & \lambda_1 & 0 & \dots 0 & 0 & 0 \\ -\lambda_1 & 0 & \lambda_2 & \dots 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & \dots 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots 0 & \lambda_{2n-1} & 0 \\ 0 & 0 & 0 & \dots -\lambda_{2n-1} & 0 & 0 \\ 0 & 0 & 0 & \dots 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ \cdot \\ a_{2n-1} \\ a_{2n} \\ a_{2n+1} \end{bmatrix}.$$

For a linear vector field X and all of the points

$P = (x_1, x_2, \dots, x_{2n+1}) \in E^{2n+1}$, we have that

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix}, \quad \text{rank } [AC] = 2n+1$$

or

$$X(P) = (\lambda_1 x_2 + a_1, -\lambda_1 x_1 + \lambda_2 x_3 + a_2, \dots, -\lambda_{2n-2} x_{2n-2} + \lambda_{2n-1} x_{2n-1} + a_{2n-1}, -\lambda_{2n-1} x_{2n-1} + a_{2n}, a_{2n+1}) .$$

In addition, if the curve

$$\alpha : I \longrightarrow E^{2n+1}$$

is an integral curve of this linear vector field X then it has to satisfy the differential equation

$$\frac{dx}{dt} = X(\alpha(t)), \ \forall t \in I \tag{III.2.1}$$

and (III.2.1) gives us the following system of differential equations

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= \lambda_1 x_2 + a_1 \\
\frac{dx_2}{dt} &= -\lambda_1 x_1 + \lambda_2 x_3 + a_2 \\
&\ \vdots \\
\frac{dx_{2n-1}}{dt} &= -\lambda_{2n-2}\, x_{2n-2} + \lambda_{2n-1}\, x_{2n} + a_{2n-1} \\
\frac{dx_{2n}}{dt} &= -\lambda_{2n-1}\, x_{2n-1} + a_{2n} \\
\frac{dx_{2n+1}}{dt} &= a_{2n+1}
\end{aligned}\right\} . \tag{III.2.2}$$

If we rewrite the matrix A by renumbering its non-zero elements λ_i we obtain that

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & A \\ 0 & 0 & B \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix} \tag{III.2.3}$$

where

$$\lambda = \begin{bmatrix} 0 & \lambda_1 & & & O \\ -\lambda_1 & 0 & \cdot & & \\ & & \cdot & & \\ & & \cdot & 0 & \lambda_m \\ O & & & -\lambda_m & 0 \end{bmatrix} \in IR^{m+1}_{m+1}, \quad C' = \begin{bmatrix} A' \\ B \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{2n+1} \end{bmatrix}$$

and $A' \in IR^{m+1}_{m+1}$, $B \in IR^{2n-m}_{1}$.

If we use the notation

$$\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$$

(III.2.1) and (III.2.3) give us that

$$\frac{dx}{dt} - \Lambda P = C' , \; C' = \begin{bmatrix} A' \\ B \end{bmatrix} \tag{III.2.4}$$

which is a first order linear differential matrix equation with constant coefficient. The solution of this equation can be given as

$$\alpha(t) = e^{t\Lambda} \left(\int_0^t C' \, e^{-u\Lambda} \, du \right) + D \tag{III.2.5}$$

or since we have that

$$e^{t\Lambda} = \begin{bmatrix} e^{t\lambda} & O \\ O & I_{2n-m} \end{bmatrix}$$

(III.2.5) can be given as the form

$$\alpha(t) = \begin{bmatrix} e^{t\lambda} \int_0^t e^{-u\lambda} \, A' \, du \\ Bt \end{bmatrix} + D \tag{III.2.6}$$

or since the matrix λ is skew-symmetric the rank λ is even and so m is odd and $\det\lambda \neq o$ and λ^{-1} is exist. Then we can write that

$$\alpha(t) = \begin{bmatrix} -\lambda^{-1} A' + \lambda^{-1} e^{t\lambda} A' \\ Bt \end{bmatrix} + D \tag{III.2.7}$$

and we give the following results:

(i) The last 2n- m components of the curve α are in the form $b_i t + d_i$.

(ii) $\|\alpha'(t)\|^2 = \|e^{t\lambda} A'\|^2 + \|B\|^2$

$= \|A'\|^2 + \|B\|^2$

this means that

$\|\alpha'(t)\| =$ constant.

Thus we can say that, in E^{2n+1}, choosing an orthonormal frame

$\{O; u_1, \ldots, u_{m+1}, u_{m+2}, \ldots, u_{2n+1}\}$

we can have that

$$\langle \alpha'(t), u_r \rangle = \|\alpha'(t)\| \cdot \|u_r\| \cdot \cos\theta_r, \; m+2 \leq r \leq 2n+1$$

$$= \sqrt{\|A'\|^2 + \|B\|^2} \quad \cos\theta_r.$$

This means that the angle in between each curve of the curves family $\alpha(t)$ and each one of the base vectors u_r is constant. Therefore, each one of the curves $\alpha(t)$ makes a constant angle with the space

$$Sp \ \{u_{m+2}, \ldots, u_{2n+2}\}.$$

On the otherhand, since the first m+1 components of each one of the curves $\alpha(t)$ can be represented by the vector

$$-\lambda^{-1}A' + \lambda^{-1} e^{t\lambda} + D_1, \ D_1 \in IR_1^{m+1}, \ D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

Using the curves

$$\tilde{\alpha}(t) = -\lambda^{-1} A' + \lambda^{-1} e^{t\lambda} A' + D$$

and the point

$$-\lambda^{-1} A' + D_1 = Q = (q_1, \ldots, q_{m+1})$$

we obtain that

$$d(Q, \tilde{\alpha}(t)) = \text{constant}.$$

Therefore we can say that:

All of the curves $\alpha(t)$ are the inclined curves and they lie on the right hypercylinders whose bases are S^m and the centres of S^m are the points

$$(q_1, q_2, \ldots, q_{m+1}, o, \ldots, o)$$

and the radiuses of those S^m are $\|\lambda^{-1} A'\|$.

So we can give the following theorem:

Theorem III.2.1. Let X be a linear vector field on E^{2n+1} determined by the matrix

$$\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$$

with respect to an orthonormal frame $\{O; u_1, u_2, \ldots, u_{2n+1}\}$, where A is a skew-symmetric matrix and C is a column matrix. Then the integral curves of this vector field X are the following ones:

(1) The integral curves of X are the inclined curves whose axises are the same.

(2) Each one of the integral curves of X lie on a rank A- dimensional right hypercylinder.

III.2.2. The Normal Form Case:

The normal form of the skew-symmetrie martrix A is

$$A = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 & 0 \dots 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 \dots 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2 & 0 \dots 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 \dots 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \dots -\lambda_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \dots 0 & 0 & 0 \end{bmatrix} \in IR^{2n+1}{}^{2n+1}, \quad \lambda \in IR-\{0\} .$$

In this case we can prove the following theorem:

Theorem III.2.2. Let X be a linear vector field in E^{2n+1} determined by the matrix

$$\begin{bmatrix} A & C \\ O & 1 \end{bmatrix}$$

with respect to an orthonoımal frame $\{O; u_1, u_2, \dots, u_{2n+1}\}$, whose A is normal formed skew-symmetric matrix, C is a column matrix. Then the integral curves of X have the following properties:

(i) If rank $[AC] = 2k + 1$, $1 \leq k \leq n$, then these curves are same parametrized circular helices which have a common axis.

(ii) If rank $[AC] = 2k$, $1 \leq k \leq n$, then those curves are circles in parallel planes whose centres lie on a same straight line perpendicular to those planes.

(iii) If rank $[AC] = 1$, then these curves are the parallel straight lines.

Proof: Let X be a lineear vector field for all points

$$P = (x_1, \dots, x_{2n+1}) \in E^{2n+1} .$$

Then we have

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix}$$

or

$X(P) = (\lambda_1 x_2 + a_1, -\lambda_1 x_1 + a_2, \ldots, \lambda_n x_{2n} + a_{2n-1}, -\lambda_n x_{n-1} + a_{2n}, a_{2n+1})$.

Moreover, if a curve

$$\alpha : I \subset IR \longrightarrow E^{2n+1}$$

is an integral curve of the vector field X, then we can write that

$$\frac{d\alpha}{dt} = X(\alpha(t)) .$$

The integral curve, with the initial condition $\alpha(t) = P$ and $P = (x_1, \ldots, x_{2n+1})$ is a solution curve of the differential equation

$$\frac{d\alpha}{dt} = X(P)$$

which means that

$$\frac{d\alpha_1}{dt} = x_2 + a_1, \frac{d\alpha_2}{dt} = -x_1 + a_2, \ldots, \frac{d\alpha_{2n-1}}{dt} = x_{2n} + a_{2n-1},$$

$\frac{d\alpha_{2n}}{dt} = -x_{2n-1} + a_{2n}, \frac{d\alpha_{2n+1}}{dt} = a_{2n+1} = C$. This means that $\lambda_i = 1$, and, $1 \leq i \leq n$.

If we solve the differential equation

$$\frac{d\alpha_{2n+1}}{dt} = C$$

we get

$$\alpha_{2n+1} = ct + d .$$

The other 2n equations can be solved in pairs. For example let us solve the first two equations

$$\frac{d\alpha_1}{dt} = x_2 + a_1, \quad \frac{d\alpha_2}{dt} = -x_1 + a_2 .$$

The general solution of these equations are

$$x_2 = A_1 \cos t + B_1 \sin t - a_1$$
$$x_1 = A_1 \sin t - B_1 \cos t + a_2 .$$

Continuing in this way, we get

$$x_{2n-1} = A_n \sin t - B_n \cos t + a_{2n}$$
$$x_{2n} = A_n \cos t + B_n \sin t - a_{2n-1} .$$

Using these solutions the expression of $\alpha(t)$ can be written as $\alpha(t) = (A_1 \sin t - B_1 \cos t + a_2,\ A_1 \cos t + B_1 \sin t - a_1, \ldots, A_n \sin t - B_n \cos t + a_{2n},\ A_n \cos t + B_n \sin t - a_{2n-1},\ ct + d)$.

Now, we show that $\alpha(t)$ is an helix in E^{2n+1} and we find the axis of it.

In order to show this, we must show that [2]

$$H_1 = \frac{k_1}{k_2} = \text{cost}.$$

The vectors $\alpha', \alpha'', \alpha'''$ and $\alpha^{(4)}, \ldots, \alpha^{(2n+1)}$ are linearly dependent. Hence there exists two curvatures k_1 and k_2. Since the curve $\alpha(t)$ does not have the unit velocity, we must apply the change of parameter to normalize the velocity.

$\alpha'(t) = (A_1 \cos t + B_1 \sin t, -A_1 \sin t + B_1 \cos t, \ldots, A_n \cos t + B_n \sin t, -A_n \sin t + B_n \cos t, C)$

and if we denote $\sum_{i=1}^{n} (A_1^2 + B_1^2)$, by Y we have

$$\|\alpha'(t)\| = \sqrt{Y + c^2},\quad s = \int_0^t \|\alpha'(t)\|\, dt$$

and if we denote $\sqrt{Y + c^2}$ by γ we get $t = \dfrac{s}{\gamma}$.

Since $E_1(s) = \beta'(s)$ and

$$<\beta'(s), \beta''(s)> = 0,\ <\beta'''(s), \beta''(s)> = 0.$$

$$E_2(s) = \beta''(s),\ E_3(s) = \frac{1}{\gamma^4} (\beta'''(s) + Y,\ E_1(s))$$

and

$$k_i(s) = \frac{\|E_{i+1}(s)\|}{\|E_i(s)\|}.$$

we have

$$H_1(s) = \sqrt{\frac{Y}{C}}.$$

This means that the curve $\beta(s)$ and so the curve $\alpha(t)$ are helices.

Let U be the axis of $\alpha(t)$. Since $U \in S_p \{V_1, V_3\}$, we have

$$U = \cos\varphi\, V_1 + \sin\varphi\, V_3 ,$$

where V_1 and V_3 are the Frenet Vector fields of the curve.

Since

$$V_1 = \frac{E_1(s)}{\|E_1(s)\|} , \; V_3 = \frac{E_3(s)}{\|E_3(s)\|} \quad \ldots\ldots (a)$$

$$H_1(s) = \sqrt{\frac{Y}{C}} \tan\varphi \text{ and } \varphi = \text{Arctan} \sqrt{\frac{Y}{C}}$$

$$U = \frac{C}{\alpha} V_1 + \frac{\sqrt{Y}}{\alpha} V_3 \quad \ldots\ldots \quad (b)$$

By combining (a) and (b) we get

$$U = (0,0,\ldots,0,1).$$

The curve $\alpha(t)$ is a cincular helix because if the curve $\alpha(t)$ is translaled by T, where

$$T = (-a_1, a_2, -a_3, a_4 - \ldots, a_{2n}, 0).$$

we obtain

$$a_1^2 + a_2^2 + \ldots + a_{2n}^2 = \sum_{i=1}^{n} (A_i^2 + B_i^2) = r = \text{constant}.$$

This completes the proof of the first part of the theorem.

2– Let rank $[AC] = 2k$, $1 \leq k \leq n$ then:

a) if rank $[AC] = 2n$, $k=n$ then differential equation system becames,

$$\frac{dx_1}{dt} = x_2 + a_1$$

$$\frac{dx_2}{dt} = -x_1 + a_2$$

$$\vdots$$

$$\frac{dx_{2n-1}}{dt} = x_{2n} + a_{2n-1}$$

$$\frac{dx_{2n}}{dt} = -x_{2n-1} + a_{2n}$$

$$\frac{dx_{2n+1}}{dt} = 0 .$$

This system of differential equations has the solution

$\alpha(t) = (A_1 \sin t - B_1 \cos t + a_2, A_1 \cos t + B_1 \sin t - a_1, \ldots, A_n \sin t - B_n \cos t + a_{2n}, A_n \cos t + B_n \sin t - a_{2n-1}, d)$.

It is trivial to show that the curves $\alpha(t)$ are circles.

b) Let rank $[AC] = r$, $r=2, \ldots, 2n-2$,

in this case

$$\text{Rank } [AC]=r \Leftrightarrow \lambda_i=0, \frac{r}{2} + 1 \leq i < n .$$

So that

$$\frac{dx_1}{dt} = x_2+a_1, \ldots, \frac{dx_{r-1}}{dt} = x_r+a_{r-1}$$

$$\frac{dx_2}{dt} = -x_1+a_2, \ldots, \frac{dx_r}{dt} = -x_r+a_r, \frac{dx_j}{dt} = 0, r+1 \leq j \leq 2n+1.$$

Therefore in this case $\alpha(t)$ is

$\alpha(t) = (A_1 \sin t - B_1 \cos t + a_2, A_1 \cos t + B_1 \sin t - a_1, \ldots, A_{r/2} \sin t - B_{r/2} \cos t + a_r, A_{r/2} \cos t + B_{r/2} \sin t - a_{r-1}, d_{r+2}, d_{r+3}, \ldots, d_{2n+1})$.

Again the curves $\alpha(t)$ are circles.

3. Rank $[AC] = 2k+1$, $1 \leq k \leq n$.

a) If rank $[AC] = 2k+1$, $k=n$,

$\alpha(t)$ is the same as the first part of the theorem.

b) Let rank $[AC] = 2k+1$

$= r+1$, $r=2,4,\ldots,2n-2$,

then rank $[AC] = r+1 \Leftrightarrow \begin{cases} \lambda_i=0 \\ a_{r+1} \neq 0 \end{cases}$, $r+1 \leq i \leq n$.

Hence $\frac{dx_1}{dt} = x_2+a_1$, $\frac{dx_2}{dt} = -x_1+a_2$,... ,

$\frac{dx_{r-1}}{dt} = x_r+a_{r-1}$, $\frac{dx_r}{dt} = -x_{r-1}+a_r$, $\frac{dx_{r+1}}{dt} = a_{r+1}$, $\frac{dx_j}{dt} = 0$

$r+2 \leq j \leq 2n+1$.

The solution of this system is

$$\alpha(t) = (A_1 \sin t - B_1 \cos t + a_2,\ A_1 \cos t + B_1 \sin t - a_1, \ldots,$$

$$A_{r/2} \sin t - B_{r/2} \cos t + a_r,\ A_{r/2} \cos t + B_{r/2} \sin t - a_{r-1},\ a_{r+1} t + d,\ d_{r+2}, \ldots, d_{2n+1}).$$

Obviously $\alpha(t)$ are again circular helices.

4. If rank $[AC] = 1$. then $\lambda_i = 0$ which gives us a system of the differential equations. This system of differential equations has the solution $\alpha(t)$ which are parallel straight lines, in E^{2n+1}.

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