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SOME SUBRING PROPERTIES OF THE RING OF HOLOMORPHIC FUNCTIONS ON A NON-EMPTY SUBSET OF AN OPEN RIEMANN SURFACE

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Throughout this paper R and S will denote open Riemann surfaces and X, Y will be non-empty subsets of R and S, respectively. A function $\varnothing: X \to S$ is said to be analytic if for each point $p \in X$ there is an open neighborhood U_p of p and an analytic function $\psi_p: U_p \to S$ such that ψ_p and $\varnothing$ coincide on $U_p \cap X$. This is equivalent to assuming that there is a single open set $U \supset X$ and an analytic function $\varnothing: U \to S$ such that $\psi \mid X = \varnothing$.Let A(X,Y) denote the set of all analytic functions $\varnothing: X \to S$ with $\varnothing(X) \subset Y$. For $Y = S = \mathbb{C}$, a function in $A(X, \mathbb{C})$ is called holomorphic and we write $H(X) = A(X,\mathbb{C})$.

It is well known that H(X) forms a ring under pointwise addition and multiplication. In fact, H(X) is an algebra over both the complex numbers $\mathbb{C}$ and the real numbers $\mathbb{R}$.

This paper is concerned with proper subrings R^* of H(X) which are isomorphic images of H(Y), the ring of all analytic functions on a non-empty subset Y of an open Riemann surface S.Φ will denote a homomorphism from H(Y) into H(X) which maps each constant function onto itself; i.e., a $\mathbb{C}$-algebra homomorphism. $\varnothing$ denotes an analytic mapping from X into $Y \subset S$; i.e., $\varnothing \in A(X,Y)$, and $R_\varnothing$ the subring of H(X) which is composed of the functions $g o \varnothing$ for $g \in H(Y)$. It has been shown that if Φ is a homomorphism of H(Y) into H(X) and Φ maps each constant function onto itself, then there is a unique analytic mapping $\varnothing \in A(X,Y)$ such that $\Phi(g) = g o \varnothing$, $g \in H(Y)$ [4]. Thus Φ (H,Y)), the image of H(Y) under Φ, is the subring $R_\varnothing$ for some analytic mapping $\varnothing \in A(X,Y)$. Now we give some basic definitions and properties of $R_\varnothing$.

A two-dimensional manifold is defined as a connected Hausdorff space M with the property that each point of M is contained in an open set homeomorphic to an open set in the Euclidean plane. The two-dimensional manifold M is an analytic manifold or abstract Riemann surface if there is a collection $\{(U_i, \theta_i): i \in I\}$ where for the index set I, $\{U_i: i \in I\}$ is an open covering of M and θ_i is a homeomorphism of U_i onto an open set in the complex plane. Also, if $U_i \cap U_j$ is non-empty, then $\theta_j \circ \theta_i^{-1}$ is a conformal sense-preserving mapping of $\theta_i (U_i \cap U_j)$ onto $\theta_j (U_i \cap U_j)$, that is $w = \theta_j \circ \theta_i^{-1} (z) = f(z)$ is an analytic function of z in $\theta_i (U_i \cap U_j)$. We say $\{(U_i, \theta_i): i \in I\}$ defines an analytic structure on the manifold M, and another collection $\{(V_j, \psi_j): j \in J\}$ defines the same analytic structure if the union of the two sets satisfies the conditions for an analytic structure on M. We say the Riemann surface is open if it is not compact.

If M is a Riemann surface, (U, θ) belongs to $\{(U_i, \theta_i): i \in I\}$ on M, p_0 belongs to U, then $z = \theta(p)$ is a local parameter about p_0 in U and there is another local parameter $w = \psi(p)$ about p_0 with $\psi(p_0) = 0$ and $|w| \leqq 1$. We define $w = (z-z_0)/r$ where $\theta(p_0) = z_0$ and $\{z: |z-z_0| \leqq r\}$ is contained in θ (U). The structure of M is not changed. A complex-valued function f on M is called analytic or holomorphic at the point p_0 if in terms of the local parameter $z=\theta(p)$, $\theta(p_0)=0$, the function $f(\theta^{-1}(z))$ is an analytic function of z for $|z| < r$ for some $r > 0$. f is holomorphic on M if f is holomorphic at each point of M. If f is a mapping of the Riemann surface M_1 into the Riemann surface M_2, $p_0 \in M_1$, $f(p_0) = q_0$, $z=\theta(p)$ is a local parameter about p_0, $w=\psi(q)$ is a local parameter about q_0, we say f is analytic on M_1 if the function $w = (f(\theta^{-1}(z)))=g(z)$ is an analytic function of z for all $p_0 \in M_1$. The two surfaces M_1 and M_2 are conformally equivalent if there is a one to-one analytic mapping of M_1 onto M_2 [1], [5].

Suppose X and Y are non-empty subsets of open Riemann surfaces R and S respectively. We define a mapping of H(Y) into H(X) by $\Phi(g)=g\circ\varnothing$ for $g \in H(Y)$. $g\circ\varnothing$ is holomorphic on X and Φ is a homomorphism. The image of Φ, $R_\varnothing = \Phi (H(Y))$ is a subring of H(X). If λ is a constant function on Y then $\Phi(\lambda) = \lambda\circ\varnothing=\lambda$ so Φ preserves constant functions. C.D. Minda proved that if R and S are open Riemann surfaces and X, Y non-empty subsets of R, S respectively, and if Φ: $H(Y) \to H(X)$ is a **C**-algebra homomorphism, then there is a unique analytic function $\varnothing$ of X into Y such that $\Phi (g) = g\circ\varnothing$ for $g \in H(Y)$ [4]. Also if Φ is an isomorphism of H(Y) into H(X), then $\varnothing$ is a one-to-one mapping of X into Y. Thus a subring R^* of H(X) is a homomorphic image of a ring H(Y) under a **C**-algebra homomorphism if and only if $R^*=R_\varnothing= \{g_1\circ\varnothing: g \in H(Y), \varnothing \in A(X, Y)\}$. $R_\varnothing$ contains the constant functions, denoted by C, since $C \subset H(Y)$ and $\Phi(\lambda) = \lambda$ for $\lambda \in C$.

A relation between Φ, $\varnothing$, and $R_{\varnothing}$ is given by the following theorem.

THEOREM 1. Let R and S be open Riemann surfaces and X, Y non-empty subsets of R, S respectively. If Φ: $H(Y) \to H(X)$ is a ring homomorphism defined by $\Phi(g) = g o \varnothing$ for $g \in H(Y)$, $\varnothing \in A(X,Y)$ and if $R_{\varnothing} = \Phi(H(Y))$, then the following three conditions are equivalent:

(a) $R_{\varnothing}$ properly contains the constant functions,

(b) $\varnothing$ is not a constant function,

(C) H(Y) is isomorphic to $R_{\varnothing}$.

Proof. Suppose $R_{\varnothing}$ properly contains C. We shall show that $\varnothing$ is not a constant function. On the contrary, if we suppose that $\varnothing(X) = \{c\}$, then $\Phi(g) = g \text{ o } \varnothing = g(c)$ for $g \in H(Y)$ which implies $R_{\varnothing} = \Phi(H(Y)) = C$. Because of this contradiction $\varnothing$ is not a constant function.

Now we shall show that (b) implies (c). Suppose that $\varnothing$ is not a constant function and $\varnothing(X)$ is a non-empty subset of Y. Let f and g be any two holomorphic functions on $\varnothing(X)$ belonging to H(Y). Then there is an open set $U \supset \varnothing(X)$ and functions F, G holomorphic on U such that $f=F \mid \varnothing(X)$, $g=G \mid \varnothing(X)$, respectively. Since f—g is holomorphic on $\varnothing(X)$, it is clear that $\Phi(f) - \Phi(g) = \Phi(f-g) = (f-g) \text{ o } \varnothing$. Thus if $\Phi(f) - \Phi(g) = 0$, then $f - g = 0$ or equivalently $f=g$. This shows that Φ is an isomorphism.

Finally we shall show that (c) implies (a). If Φ is an isomorphism, then $R_{\varnothing} \neq C$ because H(Y) contains a non-constant function g [3] and if $\Phi(g) = \lambda$, a constant function, then the set $\Phi^{-1}(\lambda)$ would contain λ and g and Φ would not be one-to-one. Thus $R_{\varnothing}$ properly contains the constant function.

COROLLARY TO THEOREM 1. A subring R^* of H(X) is isomorphic to H(Y) under a **C**-algebra isomorphism if and only if $R^* = \{g \text{ o } \varnothing : g \in H(Y), \varnothing \in A(X, Y)\}$ and R^* properly contains C the constant functions on X.

In the following theorems we shall investigate some of the relations between $R_{\varnothing}$ and $\varnothing$.

THEOREM 2. If $\varnothing$ is a one-to-one analytic mapping of X on to Y and Φ maps H(Y) into H(X) by $\Phi(g) = g \text{ o } \varnothing$, $g \in H(Y)$, then $\Phi(H(Y)) = H(X)$.

Proof. If $\varnothing$ is a one-to-one analytic mapping of X onto Y, then $\varnothing^{-1}$ is a one-to-one function from Y onto X. If $q_0 \in Y$, then $p_0 = \varnothing^{-1}(q_0) \in X$. By considering the definition of a Riemann surface and analiticity of a function between the non-empty subsets of two open Riemann surfaces we let $z = \theta(p)$, $w = \psi(g)$ be local parameters about p_0 and q_0 such that $\theta(p_0) = 0$, $\psi(q_0) = 0$. Then $\psi \circ \varnothing \circ \theta^{-1}(z)$ is analytic and one-to-one on $\{z : |z| < r_1\}$ for some $r_1 > 0$ and $\theta \circ \varnothing^{-1} \circ \psi^{-1}$ is analytic and one-to-one on $\{w: |w| < r_2\}$ for some $r_2 > 0$ since the inverse of a one-to-one analytic function is analytic. Thus $\varnothing^{-1}$ is a one-to-one analytic mapping of Y onto X. If $f \in H(X)$, then $f \circ \varnothing^{-1} \in H(Y)$ which implies $(f \circ \varnothing^{-1}) \circ \varnothing = f \in \Phi(H(Y))$. This gives us $\Phi(H(Y)) = H(X)$.

If $R_\varnothing = \Phi(H(Y))$ is to be a proper subring of H (X), then $\varnothing$ may be one-to-one or onto or neither, but not both.

THEOREM 3. Suppose $\varnothing$ is a one-to-one analytic mapping of X into Y, λ is a non-constant analytic mapping of X into Y but not one-to-one, $\Phi(g) = g \circ \varnothing$ and $\Lambda(g) = g \circ \lambda$ for $g \in H(Y)$, $R_\varnothing = \Phi(H(Y))$, $R_\lambda = \Lambda(H(Y))$. Then $R_\varnothing$ and R_λ are isomorphic but $R_\varnothing \neq R_\lambda$.

Proof. Φ and Λ are isomorphisms from H(Y) onto $R_\varnothing$ and R_λ, respectively, so $\Lambda \circ \Phi^{-1}$ is an isomorphism from $R_\varnothing$ onto R_λ. Suppose $R_\varnothing = R_\lambda$. Let $g \in H(Y)$. Then there is $h \in H(Y)$ such that $g \circ \varnothing(z) = h \circ \lambda(z)$, $z \in X$. $\varnothing$ is one-to-one and λ is not one-to-one implies there are z_1 and z_2 in X such that $z_1 \neq z_2$, $\lambda(z_1) = \lambda(z_2)$ and $\varnothing(z_1) \neq \varnothing(z_2)$. H (Y) separates the points of Y [2] implies there is $g \in H(Y)$ such that $g(\varnothing(z_1)) \neq g(\varnothing(z_2))$. But $g \circ \varnothing(z_1) = h \circ \lambda(z_1) = h \circ \lambda(z_2) = g \circ \varnothing(z_2)$. Since we reach a contradiction, $R_\varnothing \neq R_\lambda$.

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