Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (1) (2022), 218 – 227
DOI : 10.15672/hujms.897234

Research Article

Positioned numerical semigroups with maximal gender as function of multiplicity and Frobenius number

J. C. Rosales[1], M. B. Branco *[2], M. C. Faria[3]

[1] *Departamento de Álgebra, Universidad de Granada, E-18071 Granada, Spain*
[2] *Departamento de Matemática, Universidade de Évora, 7000 Évora, Portugal*
[3] *Área Departamental de Física, Instituto Sup. Eng. de Lisboa, 1959-007 Lisboa, Portugal*

Abstract

A C-semigroup (respectively a D-semigroup) is a positioned numerical semigroup S such that $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{m}(S)-1}{2}$ (respectively $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{m}(S)-2}{2}$). In this paper we study these semigroups giving formulas for the Frobenius number, pseudo-Frobenius number, and type. Furthermore, we give algorithms for computing the whole set of C-semigroups and D-semigroups.

Mathematics Subject Classification (2020). 20M14, 11D07

Keywords. numerical semigroups, positioned numerical semigroups, C-semigroups, D-semigroups tree, Frobenius number, multiplicity and gender

1. Introduction

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$. A numerical semigroup is a nonempty subset S of $\mathbb{N}$ that is closed under addition, contains the zero element, and whose complement in $\mathbb{N}$ is finite. Numerical semigroups appear in several areas of mathematics and there are several interesting combinatorial invariants of a semigroup (see for example [11]). Notable numerical semigroup invariants include the Frobenius number, multiplicity, and gender of S that are $\mathrm{F}(S) = \max\{x \in \mathbb{Z} \mid x \notin S\}$, $\mathrm{m}(S) = \min(S\backslash\{0\})$, and $\mathrm{g}(S) = \mathrm{card}(\mathbb{N}\backslash S)$, respectively.

Given a rational number q, we denote by $\lfloor q \rfloor = \max\{z \in \mathbb{Z} \mid z \leq q\}$, and $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid z \geq q\}$.

Let k be a positive integer. A numerical semigroup S is k-positioned if for all $x \in \mathbb{N}\backslash S$ we have that $k - x \in S$. The $\mathrm{F}(S)$-positioned numerical semigroups are the symmetric numerical semigroups studied in [6], [1] and [7]. The $\mathrm{F}(S) + \mathrm{m}(S)$-positioned numerical semigroups (respectively $\mathrm{F}(S) + \mathrm{m}(S) + 1$-positioned) called positioned numerical semigroups (respectively almost-positioned) are studied in [2] (respectively [3]). Thus a numerical semigroup S is positioned if for all $x \in \mathbb{N}\backslash S$ we have that $\mathrm{F}(S) + \mathrm{m}(S) - x \in S$. In [2, Proposition 5] it is shown that if S is a positioned numerical semigroup then

*Corresponding Author.
Email addresses: jrosales@ugr.es (J. C. Rosales), mbb@uevora.pt (M. B. Branco), mfaria@adf.isel.ipl.pt (M. C. Faria)
Received: 15.04.2021; Accepted: 14.09.2021

$\lceil \frac{F(S)+1}{2} \rceil \leq g(S) \leq \lfloor \frac{F(S)+m(S)-1}{2} \rfloor$. The aim of this paper is to study the positioned numerical semigroups S for which $g(S) = \lfloor \frac{F(S)+m(S)-1}{2} \rfloor$. In order to study this we distinguish two cases depending on the parity of $F(S) + m(S)$. A C-semigroup (respectively a D-semigroup) is a positioned numerical semigroup S such that $g(S) = \frac{F(S)+m(S)-1}{2}$ (respectively $g(S) = \frac{F(S)+m(S)-2}{2}$).

Let $\mathcal{A}$ be a nonempty subset of $\mathbb{N}$. We denote by $\langle \mathcal{A} \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by $\mathcal{A}$, that is,

$$\langle \mathcal{A} \rangle = \left\{ \sum_{i=1}^{n} \lambda_i \, a_i \mid n \in \mathbb{N} \backslash \{0\}, \ a_1, \dots, a_n \in \mathcal{A}, \text{and } \lambda_1, \dots, \lambda_n \in \mathbb{N} \right\}.$$

It is well known (see for example [11]) that $\langle \mathcal{A} \rangle$ is a numerical semigroup if and only if $\gcd(\mathcal{A}) = 1$.

If S is a numerical semigroup and $S = \langle \mathcal{A} \rangle$ then we say that $\mathcal{A}$ is a system of generators of S. Moreover, if $S \neq \langle \mathcal{B} \rangle$ for all $\mathcal{B} \subsetneq \mathcal{A}$, then we say that $\mathcal{A}$ is a minimal system of generators of S. In [11] it is proved that every numerical semigroup S admits a unique minimal system of generators and its cardinality is upper bounded by $m(S)$. We denote by $\mathrm{msg}(S)$ the minimal system generators of S. Its cardinality is the embedding dimension of S, denoted by $e(S)$.

A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it. In [10] it is shown that a numerical semigroup is irreducible if and only if it is either a symmetric or a pseudo-symmetric numerical semigroup. This class of semigroups is quite interesting in numerical semigroup theory (see for instance [6], [1], [5]) and there are numerous characterizations for it.

Proposition 1.1. [11, Proposition 4.4] *Let S be a numerical semigroup. S is symmetric (resp. pseudo-symmetric) if and only if $F(S)$ is odd (resp. even) and $x \in \mathbb{Z} \backslash S$ implies $F(S) - x \in S$ (resp. $x \in \mathbb{Z} \backslash S$ implies that either $F(S) - x \in S$ or $x = \frac{F(S)}{2}$).*

Given a numerical semigroup S, we say that an integer x is a pseudo-Frobenius number if $x \in \mathbb{Z} \backslash S$ and $x + s \in S$ for all $s \in S \backslash \{0\}$. We denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S, and its cardinality is the type of S, denoted by $t(S)$.

In section 2 we show that S is a C-semigroup if and only if $S = S' \cup \{F(S')\}$ with S' a symmetric numerical semigroup different from $\mathbb{N}$ and $\langle \{2, 3\} \rangle$. We will also give formulas for Frobenius number, pseudo-Frobenius numbers and type of a C-semigroup in terms of its minimal system of generators.

Let S be a numerical semigroup and $\mathrm{msg}(S) = \{n_1, \dots, n_p\}$. We say that an element $s \in S$ has unique expression if there exists a unique n-tuple $(\lambda_1, \dots, \lambda_p) \in \mathbb{N}^p$ such that $s = \lambda_1 n_1 + \cdots + \lambda_p n_p$.

In section 3 we see that S is a D-semigroup if and only if $\frac{F(S)+m(S)}{2} \in \mathrm{msg}(S)$ and $F(S) + m(S)$ has unique expression in S. The same way, as before, we show that S is a D-semigroup if and only if $S = S' \cup \{F(S'), \frac{F(S')}{2}\}$ with S' a pseudo-symmetric numerical semigroup with $F(S') > 2m(S')$. Again, we will also give formulas for the Frobenius number, pseudo-Frobenius numbers and type of a D-semigroup in terms of its minimal system of generators.

Finally, in section 4, the study done in the previous sections along with [4] allows us to give a procedure to compute the whole set of C-semigroups and D-semigroups with given Frobenius number and multiplicity.

2. C-semigroups

Recall that a C-semigroup is a positioned numerical semigroup S with $g(S) = \frac{F(S)+m(S)-1}{2}$. Our aim in this section is to characterize this type of numerical semigroups.

A numerical semigroup $\{0, \mathrm{m}(S) \rightarrow\}$ is sometimes called in the literature half-line or ordinary and it will be denoted here by $\Delta(m)$.

Proposition 2.1. *If $m \in \mathbb{N}\backslash\{0,1\}$, then $\Delta(m)$ is a C-semigroup.*

Proof. The half-line $\Delta(m)$ is a positioned numerical semigroup with $\mathrm{m}(\Delta(m)) = m$, $\mathrm{F}(\Delta(m)) = m-1$ and $\mathrm{g}(\Delta(m)) = m-1$. Then we deduce that $\mathrm{g}(\Delta(m)) = \frac{\mathrm{F}(\Delta(m))+\mathrm{m}(\Delta(m))-1}{2}$ and thus $\Delta(m)$ is a C-semigroup. □

Given a numerical semigroup S, denote by

$$Q(S) = \{x \in S \mid 1 \leq x \leq \mathrm{F}(S) + \mathrm{m}(S) - 1\}.$$

Its cardinality is denoted by $\mathrm{q}(S)$.

It is easy to prove our next result which is deduced in [2, Proposition 5].

Lemma 2.2. *Let S be a positioned numerical semigroup. Then*

(1) *$\phi : \mathbb{N}\backslash S \rightarrow Q(S)$, defined by $\phi(x) = \mathrm{F}(S) + \mathrm{m}(S) - x$ is an injective map.*
(2) *$\mathrm{g}(S) \leq \mathrm{q}(S)$.*
(3) *S is a C-semigroup if and only if $\mathrm{g}(S) = \mathrm{q}(S)$.*

Note that if $m \in \mathrm{msg}(S)$ and $x \in S\backslash\{0, m\}$ then $m - x \notin S$ and so we have the following result.

Lemma 2.3. *Let S be a numerical semigroup and $m \in \mathrm{msg}(S)$. If $x \in \{1, \ldots, m-1\} \cap S$ then $m - x \in \{1, \ldots, m-1\} \cap \mathbb{N}\backslash S$.*

From the previous lemma we can state the next result.

Lemma 2.4. *Let S be a numerical semigroup such that $\mathrm{F}(S) + \mathrm{m}(S) \in \mathrm{msg}(S)$. Then the following conditions hold:*

(1) *$\psi : Q(S) \rightarrow \mathbb{N}\backslash S$, defined by $\psi(x) = \mathrm{F}(S) + \mathrm{m}(S) - x$ is a injective map.*
(2) *$\mathrm{q}(S) \leq \mathrm{g}(S)$.*

Theorem 2.5. *Let S be a positioned numerical semigroup. Then S is a C-semigroup if and only if $\mathrm{F}(S) + \mathrm{m}(S) \in \mathrm{msg}(S)$.*

Proof. *Necessity*. If $\mathrm{F}(S) + \mathrm{m}(S) \notin \mathrm{msg}(S)$, then there exists $\{x, y\} \subseteq S\backslash\{0\}$ such that $\mathrm{F}(S) + \mathrm{m}(S) = x + y$. By Lemma 2.2, we deduce that ϕ is not a surjective map and so $\mathrm{g}(S) < \mathrm{q}(S)$. By using again (3) of the same lemma, we have that S is not a C-semigroup.

Sufficiency. By using Lemmas 2.2 and 2.4, we conclude that $\mathrm{g}(S) \leq \mathrm{q}(S)$ and $\mathrm{q}(S) \leq \mathrm{g}(S)$. Hence, $\mathrm{g}(S) = \mathrm{q}(S)$ and by (3) of Lemma 2.2 we obtain that S is a C-semigroup. □

Given a numerical semigroup S, we denote by $M(S) = \max(\mathrm{msg}(S))$. It is easy to prove the following result.

Lemma 2.6. *Let S be a numerical semigroup. If $\mathrm{F}(S) + \mathrm{m}(S) \in \mathrm{msg}(S)$, then $M(S) = \mathrm{F}(S) + \mathrm{m}(S)$.*

The following result is well-known and appears in [11].

Lemma 2.7. *Let S be a numerical semigroup and let $x \in S$. Then $S\backslash\{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*

Proposition 2.8. *If S is a C-semigroup, then $\overline{S} = S\backslash\{M(S)\}$ is a symmetric numerical semigroup with $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$ and $\mathrm{F}(\overline{S}) = \mathrm{F}(S) + \mathrm{m}(S)$.*

Proof. Using Theorem 2.5 and Lemma 2.6, we obtain that $M(S) = \mathrm{F}(S) + \mathrm{m}(S)$. By Lemma 2.7, we get that $\overline{S}$ is numerical semigroup with $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$ and $\mathrm{F}(\overline{S}) = \mathrm{F}(S) + \mathrm{m}(S)$.

In order to prove that $\overline{S}$ is symmetric, we will see that if $x \in \mathbb{N}\backslash\overline{S}$ then $\mathrm{F}(\overline{S}) - x \in \overline{S}$. If $x \in \mathbb{N}\backslash\overline{S}$, then either $x \in \mathbb{N}\backslash S$ or $x = M(S)$. We distinguish two cases.

- If $x = M(S)$, then $\mathrm{F}(\overline{S}) - x = M(S) - M(S) = 0 \in \overline{S}$
- If $x \in \mathbb{N}\backslash S$, then $\mathrm{F}(S) + \mathrm{m}(S) - x \in S$ and so $\mathrm{F}(S) + \mathrm{m}(S) - x \in \overline{S}$ or $\mathrm{F}(S) + \mathrm{m}(S) - x = M(S)$. If $\mathrm{F}(S) + \mathrm{m}(S) - x = M(S)$, then $x = 0$, which contradicts $x \in \mathbb{N}\backslash\overline{S}$. Hence $\mathrm{F}(\overline{S}) - x \in \overline{S}$ and thus $\overline{S}$ is symmetric.

□

The next result it easy to prove.

Lemma 2.9. *If S is a numerical semigroup such that $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is again a numerical semigroup.*

The following result can be deduce of [9, Lemmas 1.2 and 1.3].

Lemma 2.10. *Let S be a symmetric numerical semigroup such that $S \notin \{\mathbb{N}, \langle 2,3\rangle\}$ and let $\overline{S} = S \cup \{\mathrm{F}(S)\}$. Then $\overline{S}$ is a numerical semigroup with $\mathrm{F}(\overline{S}) = \mathrm{F}(S) - \mathrm{m}(S)$ and $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$.*

Proposition 2.11. *If S is a symmetric numerical semigroup such that $S \neq \mathbb{N}$, then $\overline{S} = S \cup \{\mathrm{F}(S)\}$ is a positioned numerical semigroup.*

Proof. We distinguish two cases.

- If $S = \langle 2, 3\rangle$ then $\overline{S} = \mathbb{N}$ and thus $\overline{S}$ is a positioned numerical semigroup.
- If $S \neq \langle 2, 3\rangle$, then by Lemma 2.10, we have that $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$ and $\mathrm{F}(\overline{S}) + \mathrm{m}(\overline{S}) = \mathrm{F}(S)$. Whence, if $x \in \mathbb{N}\backslash\overline{S}$, then $x \in \mathbb{N}\backslash S$ and so $\mathrm{F}(S) - x \in S$. Consequently, $\mathrm{F}(\overline{S}) + \mathrm{m}(\overline{S}) - x \in \overline{S}$ and thus $\overline{S}$ is a positioned numerical semigroup.

□

A numerical semigroup S is called a UESYsemigroups if there exists S' symmetric such that $S = S' \cup \{\mathrm{F}(S')\}$.

Proposition 2.12. [9, Theorem 1.8] *Let S be a numerical semigroup such that $S \neq \mathbb{N}$. The following conditions are equivalent.*

(1) *S is a UESYsemigroup.*
(2) *$\mathrm{F}(S) + \mathrm{m}(S) \in \mathrm{msg}(S)$ and $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{m}(S)-1}{2}$.*

Theorem 2.13. *Let S be a numerical semigroup such that $S \neq \mathbb{N}$. Then S is a C-semigroup if and only if S is a UESYsemigroup.*

Proof. *Necessity.* If S is a C-semigroup, then $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{m}(S)-1}{2}$. By Theorem 2.5, we have that $\mathrm{F}(S) + \mathrm{m}(S) \in \mathrm{msg}(S)$. Hence, by applying Proposition 2.12, we obtain that S is a UESYsemigroup.

Sufficiency. If S is a UESYsemigroup then there exist a symmetric numerical semigroup S' such that $S = S' \cup \{\mathrm{F}(S')\}$ with $S' \notin \{\langle 2, 3\rangle, \mathbb{N}\}$. By Propositions 2.11 and 2.12, we have that S is a positioned semigroup and $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{m}(S)-1}{2}$. Whence S is a C-semigroup. □

By using Theorem 2.13 and [9, Corollary 1.9 and Theorem 1.14] we obtain the following result.

Proposition 2.14. *Let S be a C-semigroup . Then the following conditions hold.*

(1) *$\mathrm{F}(S) = M(S) - \mathrm{m}(S)$.*
(2) *$\mathrm{g}(S) = \frac{M(S)-1}{2}$.*
(3) *$\mathrm{PF}(S) = \{M(S) - x \mid x \in \mathrm{msg}(S)$ and $x \neq M(S)\}$ and $\mathrm{t}(S) = \mathrm{e}(S) - 1$.*

Note that $\mathbb{N}$ is the unique numerical semigroup with embedding dimension one which is not C-semigroup and so there are no C-semigroups with embedding dimension one.

In [11] it is proved that, if S is a numerical semigroup with $\mathrm{e}(S) = 2$, then S is symmetric and $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$. We deduce that if S is a C-semigroup with $\mathrm{e}(S) = 2$, then this forces

$\mathrm{m}(S) = 2$. Since all symmetric numerical semigroups are positioned (see [2]) we obtain the next result.

Proposition 2.15. *The set of C-semigroups with embedding dimension two is equal to $\{\langle 2, 2k+1\rangle \mid k \in \mathbb{N}\backslash\{0\}\}$.*

It is well known that if S is a numerical semigroup, then $\mathrm{e}(S) \leq \mathrm{m}(S)$ and $\mathrm{t}(S) \leq m-1$ (see [11]). The next aim, in this section, is to show that for given m and e integers (resp. m and t integers) such that $3 \leq e \leq m$ (resp. $2 \leq t \leq m-1$) there exists a C-semigroup with multiplicity m and embedding dimension e (resp. with multiplicity m and type t).

The next lemmas are known and they are in [9].

Lemma 2.16. [9, Lemma 1.13] *Let S be a symmetric numerical semigroup with $\mathrm{m}(S) \geq 3$. Then $\mathrm{e}(S \cup \{\mathrm{F}(S)\}) = \mathrm{e}(S)+1$ and $\mathrm{t}(S \cup \{\mathrm{F}(S)\}) = \mathrm{e}(S)$.*

Lemma 2.17. [9, Lemma 1.16] *Let m and e be integers such that $2 \leq e \leq m-1$. Then there exists a symmetric numerical semigroup S with $\mathrm{m}(S) = m$ and $\mathrm{e}(S) = e$.*

Proposition 2.18. *Let m and e be integers such that $3 \leq e \leq m$. Then there exists a C-semigroup S with $\mathrm{m}(S) = m$ and $\mathrm{e}(S) = e$.*

Proof. By Lemma 2.17, there exists a symmetric numerical semigroup S' with $\mathrm{m}(S') = m$ and $\mathrm{e}(S') = e-1$. Suppose that $S = S' \cup \{\mathrm{F}(S')\}$. Then S is a UESYsemigroup and, by Theorem 2.13, is a C-semigroup. Further, by Lemmas 2.10 and 2.16, we obtain that $\mathrm{m}(S) = m$ and $\mathrm{e}(S) = e$. □

Remark 2.19. Observe that S is a symmetric numerical semigroup if and only if $\mathrm{t}(S) = 1$. Besides, by the previous comment to Proposition 2.15 about the multiplicity, we obtain that the set of C-semigroups with type one is equal to $\{\langle 2, 2k+1\rangle \mid k \in \mathbb{N}\backslash\{0\}\}$.

Proposition 2.20. *Let m and t be integers such that $2 \leq t \leq m-1$. Then there exists a C-semigroup S with $\mathrm{m}(S) = m$ and $\mathrm{t}(S) = t$.*

Proof. By Lemma 2.18, there exists a C-semigroup S with $\mathrm{m}(S) = m$ and $\mathrm{e}(S) = t+1$. By applying Proposition 2.14, $\mathrm{t}(S) = t$. □

Example 2.21. The numerical semigroup $S' = \langle 7, 11\rangle$ is a symmetric numerical semigroup with $\mathrm{F}(S') = 59$. Hence $S' \cup \{\mathrm{F}(S')\} = \langle 7, 11, 59\rangle$ is a UESYsemigroup and, by Theorem 2.13, is a C-semigroup. Using Proposition 2.14, we have that $\mathrm{F}(S' \cup \{\mathrm{F}(S')\}) = 59-7 = 52$, $\mathrm{g}(S' \cup \{\mathrm{F}(S')\}) = \frac{59-1}{2} = 29$, $\mathrm{PF}(S' \cup \{\mathrm{F}(S')\}) = \{59-7, 59-11\} = \{52, 48\}$ and $\mathrm{t}(S' \cup \{\mathrm{F}(S')\}) = 2$.

3. *D*-semigroups

Recall that a D-semigroup is a positioned numerical semigroup S with $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{m}(S)-2}{2}$. Our aim in this section is to characterize these types of numerical semigroups.

Example 3.1. Let us see that $S = \langle 6, 7, 11\rangle$ is a D-semigroup.
In fact $S = \{0, 6, 7, 11, 12, 13, 14, 17, \rightarrow\}$ and thus $\mathrm{F}(S) = 16$, $\mathrm{m}(S) = 6$,
$\mathbb{N}\backslash S = \{1, 2, 3, 4, 5, 8, 9, 10, 15, 16\}$ and $\mathrm{g}(S) = 10$.
Moreover $\{\mathrm{F}(S) + \mathrm{m}(S) - x \mid x \in \mathbb{N}\backslash S\} = \{21, 20, 19, 18, 17, 14, 13, 12, 7, 6\} \subseteq S$ and so S is positioned. Since $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{m}(S)-2}{2}$, we obtain that S is a D-semigroup

Lemma 3.2. *If S is a positioned numerical semigroup and $\mathrm{F}(S)+\mathrm{m}(S)$ is an even number, then $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in S$.*

Proof. Since S is positioned, if $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in \mathbb{N}\backslash S$ then $\mathrm{F}(S) + \mathrm{m}(S) - \frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in S$. Hence $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in S$, a contradiction. □

Lemma 3.3. *Let S be a positioned numerical semigroup. Then S is a D-semigroup if and only if* $\mathrm{g}(S) = \mathrm{q}(S) - 1$.

Proof. If S be a positioned numerical semigroup, then $\mathrm{g}(S) + \mathrm{q}(S) = \mathrm{F}(S) + \mathrm{m}(S) - 1$. Hence, we obtain that S is a D-semigroup (i.e. $2\mathrm{g}(S) = \mathrm{F}(S) + \mathrm{m}(S) - 2$) if and only if $\mathrm{g}(S) = \mathrm{q}(S) - 1$. □

Lemma 3.4. *If S is a D-semigroup, then* $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in \mathrm{msg}(S)$ *and* $\mathrm{F}(S) + m(S)$ *has unique expression in S.*

Proof. By Lemma 3.2, we have that $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in S$. Since S is a D-semigroup, then $S \neq \mathbb{N}$ and so $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \neq 0$. If $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \notin \mathrm{msg}(S)$, then there exist $x, y \in S\backslash\{0\}$ such that $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} = x + y$. Hence $\mathrm{F}(S) + \mathrm{m}(S) = 2x + 2y = x + (x + 2y) = (2x + y) + y$. Now, by using Lemma 2.2, we obtain that $\{x, 2x, 2x + y\} \subseteq Q(S)$ and $\{x, 2x, 2x + y\} \cap \phi(\mathbb{N}\backslash S) = \emptyset$ and thus $\mathrm{q}(S) \geq \mathrm{g}(S) + 3$.

If $\mathrm{F}(S) + \mathrm{m}(S)$ has not unique expression in S, then there exist $x, y \in S\backslash\{0\}$ such that $\mathrm{F}(S) + \mathrm{m}(S) = x + y$ and $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \notin \{x, y\}$. By Lemma 2.2 again, we get that $\mathrm{q}(S) \geq \mathrm{g}(S) + 2$.

In both cases, we have that if $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \notin \mathrm{msg}(S)$ or if $\mathrm{F}(S) + \mathrm{m}(S)$ does not have unique expression in S, then $\mathrm{g}(S) \neq \mathrm{q}(S) - 1$. But this contradicts Lemma 3.3, since S is a D-semigroup. □

Theorem 3.5. *Let S be a positioned numerical semigroup. Then S is a D-semigroup if and only if* $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in \mathrm{msg}(S)$ *and* $\mathrm{F}(S) + \mathrm{m}(S)$ *has unique expression in S.*

Proof. *Necessity.* This is an immediate consequence of the Lemma 3.4.

Sufficiency. From the hypothesis, we deduce that $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2}$ is the unique element in $Q(S)\backslash\mathrm{Im}(\phi)$ (in view of of Lemma 2.2) and thus $\mathrm{g}(S) = \mathrm{q}(S) - 1$. By using Lemma 3.3, we have that S is a D-semigroup. □

Lemma 3.6. *If S is a D-semigroup, then*

$$\overline{S} = S\backslash\left\{\frac{\mathrm{F}(S) + \mathrm{m}(S)}{2}, \mathrm{F}(S) + \mathrm{m}(S)\right\}$$

is a numerical semigroup.

Proof. As by Theorem 3.5, we have that $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in \mathrm{msg}(S)$, then, by applying Lemma 2.7, $S' = S\backslash\left\{\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2}\right\}$ is a numerical semigroup. Using again Theorem 3.5, $\mathrm{F}(S)+\mathrm{m}(S)$ has unique expression in S. We can deduce that $\mathrm{F}(S) + \mathrm{m}(S) \in \mathrm{msg}(S')$ and in the same way $\overline{S} = S'\backslash\{\mathrm{F}(S) + \mathrm{m}(S)\}$ is a numerical semigroup. □

Proposition 3.7. *If S is a D-semigroup, then*

$$\overline{S} = S\backslash\left\{\frac{\mathrm{F}(S) + \mathrm{m}(S)}{2}, \mathrm{F}(S) + \mathrm{m}(S)\right\}$$

is a pseudosymmetric numerical semigroup with $2\mathrm{m}(\overline{S}) < \mathrm{F}(\overline{S})$.

Proof. Obviously $\mathrm{F}(\overline{S}) = \mathrm{F}(S) + \mathrm{m}(S)$. We need to show that $\overline{S}$ is a pseudosymmetric numerical semigroup (i.e. $x \in \mathbb{Z}\backslash\overline{S}$ implies that either $\mathrm{F}(\overline{S}) - x \in \overline{S}$ or $x = \frac{\mathrm{F}(\overline{S})}{2}$).

We have that if $x \in \mathbb{N}\backslash\overline{S}$ then either $x \in \mathbb{N}\backslash S$ or $x = \frac{\mathrm{F}(S)+\mathrm{m}(S)}{2}$ or $x = \mathrm{F}(S) + \mathrm{m}(S)$. We distinguish three cases.

- If $x \in \mathbb{N}\backslash S$ then $\mathrm{F}(S) + \mathrm{m}(S) - x \in S$ and so $\mathrm{F}(\overline{S}) - x \in S$. Hence either $\mathrm{F}(\overline{S}) - x \in \overline{S}$ or $\mathrm{F}(\overline{S}) - x = \frac{F(\overline{S})}{2}$ or $\mathrm{F}(\overline{S}) - x = \mathrm{F}(\overline{S})$. As the last two cases are not possible, we get that $\mathrm{F}(\overline{S}) - x \in \overline{S}$.

- If $x = \frac{\mathrm{F}(S)+\mathrm{m}(S)}{2}$, then $2x = \mathrm{F}(\overline{S})$, contradicting that $2x \neq \mathrm{F}(\overline{S})$.
- If $x = \mathrm{F}(S) + \mathrm{m}(S)$, then $x = \mathrm{F}(\overline{S})$ and $\mathrm{F}(\overline{S}) - x = \mathrm{F}(\overline{S}) - \mathrm{F}(\overline{S}) = 0 \in \overline{S}$.

Since $\mathrm{F}(S) + \mathrm{m}(S)$ is even, then $\mathrm{F}(S) \neq \mathrm{m}(S) - 1$ and so $\mathrm{F}(S) > \mathrm{m}(S)$. Therefore $\mathrm{m}(S) = \mathrm{m}(\overline{S})$ and $\mathrm{F}(\overline{S}) > 2\mathrm{m}(\overline{S})$. □

As a consequence of [8, Lemmas 31 and 32] we obtain the following.

Lemma 3.8. *Let S be a pseudosymmetric numerical semigroup with $\mathrm{F}(S) > 2\mathrm{m}(S)$. Then $\overline{S} = S \cup \{\frac{\mathrm{F}(S)}{2}, \mathrm{F}(S)\}$ is a numerical semigroup, $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$ and $\mathrm{F}(\overline{S}) = \mathrm{F}(S) - \mathrm{m}(S)$*

Proposition 3.9. *Let S be a pseudosymmetric numerical semigroup with $\mathrm{F}(S) > 2\mathrm{m}(S)$. Then $\overline{S} = S \cup \{\frac{\mathrm{F}(S)}{2}, \mathrm{F}(S)\}$ is a positioned numerical semigroup.*

Proof. From Lemma 3.8 we obtain that $\overline{S}$ is a numerical semigroup, $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$ and $\mathrm{F}(\overline{S}) = \mathrm{F}(S) - \mathrm{m}(S)$. We need to see that $\overline{S}$ is positioned. If $x \in \mathbb{N}\backslash\overline{S}$ then $x \in \mathbb{N}\backslash S$ and $x \neq \frac{\mathrm{F}(S)}{2}$. Since S is pseudosymmetric, we obtain that $\mathrm{F}(S) - x \in S \subseteq \overline{S}$. Hence $\mathrm{F}(\overline{S}) + \mathrm{m}(\overline{S}) - x \in \overline{S}$ and so $\overline{S}$ is positioned. □

A numerical semigroup S is called PEPSY-semigroup if there exist S' pseudosymmetric numerical semigroup such that $S = S' \cup \left\{F(S'), \frac{\mathrm{F}(S')}{2}\right\}$.

A PEPSY-semigroup that is not a half-line is called PEPSYNHL-semigroup.

From this definitions we have the following results of [8].

Lemma 3.10. [8, Proposition 30] *A numerical semigroup S is a PEPSY-semigroup if and only if one of the following conditions holds:*

(1) *S is half-line.*
(2) *there exists a pseudo-symmetric numerical semigroup S' with $F(S') > 2\mathrm{m}(S')$ and $S = S' \cup \left\{F(S'), \frac{\mathrm{F}(S')}{2}\right\}$.*

Lemma 3.11. [8, Theorem 33] *Let S not be half-line. The following conditions are equivalent:*

(1) *S is a PEPSY-semigroup,*
(2) *$\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in \mathrm{msg}(S)$, $\mathrm{F}(S)+\mathrm{m}(S)$ has unique expression in S and $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{m}(S)-2}{2}$.*

Theorem 3.12. *A semigroup S is a D-semigroup if and only if S is a PEPSYNHL-semigroup.*

Proof. *Necessity*. If S is a D-semigroup, then we have that $\mathrm{F}(S) + \mathrm{m}(S)$ is even and thus S is not half-line. Besides, we have that $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{m}(S)-2}{2}$ and, by Theorem 3.5, $\frac{\mathrm{F}(S)+\mathrm{m}(S)}{2} \in \mathrm{msg}(S)$ and $\mathrm{F}(S) + \mathrm{m}(S)$ has unique expression in S. Hence we conclude, by Lemma 3.11, that S is a PEPSYNHL-semigroup.

Sufficiency. Suppose that S is a PEPSYNHL-semigroup. By using Lemma 3.10, there exists a pseudo-symmetric numerical semigroup S' with $F(S') > 2\mathrm{m}(S')$ and $S = S' \cup \left\{F(S'), \frac{\mathrm{F}(S')}{2}\right\}$. By Lemma 3.8 and Proposition 3.9, we get that S is positioned and $\mathrm{m}(S') = \mathrm{m}(S)$ and $\mathrm{F}(S) = \mathrm{F}(S') - \mathrm{m}(S')$. As S' is pseudo-symmetric then $\mathrm{g}(S') = \frac{\mathrm{F}(S')+2}{2}$ and $\mathrm{g}(S) = \mathrm{g}(S') - 2$. Therefore, we can conclude that $\mathrm{g}(S) = \frac{\mathrm{F}(S')+2}{2} - 2 = \frac{\mathrm{F}(S)+\mathrm{m}(S)-2}{2}$ and so S is a D-semigroup. □

As a consequence of Lemma 3.10 and Theorem 3.12 we obtain the following result.

Corollary 3.13. *The set of all D-semigroups is equal to*

$$\left\{ S' \cup \{\mathrm{F}(S'), \frac{\mathrm{F}(S')}{2}\} \mid S' \text{ pseudo-symmetric semigroup with } \mathrm{F}(S') > 2\mathrm{m}(S') \right\}.$$

The next results give us formulas for the Frobenius number and pseudo-Frobenius numbers of a D-semigroup. The first (resp. second) is a consequence of Theorem 3.12 and [8, Corollary 34] (resp. Theorem 3.12 and [8, Lemma 35 and Theorem 38]).

Proposition 3.14. *If S is a D-semigroup, then $\mathrm{F}(S) \leq 2M(S) - \mathrm{m}(S)$. Moreover, $\mathrm{F}(S) = 2x - \mathrm{m}(S)$ for some $x \in \mathrm{msg}(S)$ such that $2x > M(S)$ and $2x$ has unique expression in S.*

Proposition 3.15. *If S is a D-semigroup, then $\mathrm{t}(S) = \mathrm{e}(S) - 1$. Moreover, $\mathrm{PF}(S) = \left\{\mathrm{F}(S) + \mathrm{m}(S) - x \mid x \in \mathrm{msg}(S) \text{ and } x \neq \frac{\mathrm{F}(S)+\mathrm{m}(S)}{2}\right\}$.*

4. The algorithm

Given positive integers m and F, we denote by

$$P(m, F) = \{S \mid S \text{ is a positioned semigroup with } \mathrm{m}(S) = m, \text{ and } \mathrm{F}(S) = F\},$$

$$\text{and} \quad \mathfrak{P}(m, F) = \left\{S \in P(m, F) \mid \mathrm{g}(S) = \lfloor \frac{F + m - 1}{2} \rfloor \right\}.$$

The aim of this section is to show how to construct an algorithm to compute all elements in $\mathfrak{P}(m, F)$.
Clearly, if $C(m, F) = \{S \in P(m, F) \mid S \text{ is a } C\text{-semigroup}\}$ and $D(m, F) = \{S \in P(m, F) \mid S \text{ is a } D\text{-semigroup}\}$, then

$$\mathfrak{P}(m, F) = \begin{cases} C(m, F) & \text{if } F + m \text{ is odd} \\ D(m, F) & \text{if } F + m \text{ is even}. \end{cases}$$

4.1. Case $m + F$ odd

Since $\mathrm{g}(\mathbb{N}) = 0$ we have that $\mathbb{N}$ is not a C-semigroup. By Remark 2.19, we deduce that the whole set of C-semigroups with $\mathrm{m}(S) = 2$ is equal to $\{\langle 2, 2k + 1\rangle \mid k \in \mathbb{N}\backslash\{0\}\}$. Since $\mathrm{F}(\langle 2, 2k + 1\rangle) = 2k - 1$, then F is an odd integer if and only if $C(2, F) \neq \emptyset$. Therefore, we can conclude that $C(2, F) = \{\langle 2, F + 2\rangle\}$.
From now on we assume that $m \geq 3$. Clearly, if S is a numerical semigroup and $\mathrm{m}(S) = m$, then $\mathrm{F}(S) \geq m - 1$. If $\mathrm{F}(S) = m - 1$ then $S = \Delta(m)$ is a half-line, and by Proposition 1, it is C-semigroup. Therefore, we have that $C(m, m - 1) = \{\Delta(m)\}$.
So let us assume that $3 \leq m < F$ and $F + m$ is odd. From Lemma 2.10 and Theorem 2.13 we deduce the next result.

Proposition 4.1. *With the above notation, we have that*

$$C(m, F) = \{S \cup \{\mathrm{F}(S)\} \mid S \text{ is a symmetric numerical semigroup}, \mathrm{m}(S) = m \text{ and } \mathrm{F}(S) = F + m\}.$$

We denote by $\mathfrak{I}(m, F)$ the set of all irreducible numerical semigroups with multiplicity m and Frobenius number F. Recall that a numerical semigroup is symmetric (respectively pseudo-symmetric) if it is irreducible with Frobenius number odd (respectively even).

Lemma 4.2. [4, Proposition 13] *Let m and F be integers such that $F \geq 3$. Then $\mathfrak{I}(m, F) \neq \varnothing$ if and only if $m \leq \frac{F+2}{2}$ and $m \nmid F$.*

As a consequence of Proposition 4.1 and Lemma 4.2 we have the following.

Proposition 4.3. *With the above notation, we have that $C(m, F) \neq \varnothing$ if and only if $m \nmid F$.*

Now, we are able to give an algorithm to compute the whole set $C(m, F)$.

Algorithm 1
INPUT: Integeres m and F such that $3 \leq m < F$, $F + m$ is odd and $m \nmid F$.
OUTPUT: The set $C(m, F)$.

1: Compute the set $\mathfrak{I}(m, F)$ applying [4, Algorithm 22]
2: Return $C(m, F) = \{S \cup \{F + m\} \mid S \in \mathfrak{I}(m, F + m)\}$.

Example 4.4. Let us compute the whole set $C(6, 13)$.

(1) Compute the set $\mathfrak{I}(6, 19)$ applying [4, Algorithm 22]. We start by the root of the tree $\langle 6, 10, 11, 14, 15\rangle$ and we obtain that $\mathfrak{I}(6, 19) = \{\langle 6, 10, 11, 14, 15\rangle, \langle 6, 8, 10, 15, 17\rangle, \langle 6, 9, 11, 14, 16\rangle, \langle 6, 8, 9\rangle\}$.
(2) $C(6, 13) = \{\langle 6, 10, 11, 14, 15, 19\rangle, \langle 6, 8, 10, 15, 17, 19\rangle, 6, 9, 11, 14, 16, 19\rangle, \langle 6, 8, 9, 19\}$.

4.2. Case $m + F$ even

From previous results we have that there are no D-semigroups with multiplicity 1 and 2. On the other hand, as a half-line is not a D-semigroup we obtain that if S is a D-semigroup then $\mathrm{F}(S) > \mathrm{m}(S)$. So let us assume that $3 \leq m < F$ and $F + m$ is even.

As a consequence of Theorem 3.12 and Lemma 3.8 we have the following.

Proposition 4.5. *With the above notation, we have that*

$$D(m, F) = \Big\{S \cup \{F + m, \frac{F + m}{2}\} \mid S \textit{ is a pseudosymmetric numerical semigroup}, \mathrm{m}(S) = m \textit{ and } \mathrm{F}(S) = F + m\Big\}.$$

From Lemma 4.2 and Proposition 4.5 we deduce the next result.

Proposition 4.6. *With the above notation, we have that $D(m, F) \neq \varnothing$ if and only if $m \nmid F$.*

We conclude by giving an algorithm that will allow us to compute the whole set $D(m, F)$

Algorithm 2
INPUT: Integeres m and F such that $3 \leq m < F$, $F + m$ is even and $m \nmid F$.
OUTPUT: The set $D(m, F)$.

1: Compute the set $\mathfrak{I}(m, F)$ applying [4, Algorithm 22]
2: Return $D(m, F) = \Big\{S \cup \{F + m, \frac{F+m}{2}\} \mid S \in \mathfrak{I}(m, F + m)\Big\}$.

Acknowledgment. We thank the anonymous referees for their detailed suggestions and comments, which have greatly improved this article.

References

[1] V. Barucci, D.E. Dobbs and M. Fontana, *Maximality Properties in Numerical Semigroups and Applications to One-Dimensional Analytically Irreducible Local Domains*, Memoirs of the Amer. Math. Soc. 598, 1997.

[2] M.B. Branco, M.C. Faria and J.C. Rosales, *Positioned numerical semigroups*, Quaest. Math. **44**, 679-691, 2021.

[3] M.B. Branco, M.C. Faria and J.C. Rosales, *Almost-positioned numerical semigroups*, Results Math. **76**, 1-14, 2021.

[4] M.B. Branco, I. Ojeda and J.C. Rosales, *The set of numerical semigroups of a given multiplicity and Frobenius number*, Port. Math. **78**, 147-167, 2021.

[5] R. Fröberg, C. Gottlieb and R. Häggkvist, *On numerical semigroups*, Semigroup Forum **35**, 63-83, 1987.

[6] E. Kunz, *The value-semigroup of a one-dimensional Gorenstein ring*, Proc. Amer. Math. Soc. **25**, 748-751, 1973.

[7] J.C. Rosales, On symmetric numerical semigroups, J. Algebra **182**, 422-434, 1996.

[8] J.C. Rosales, *Adding or removing on element from a Pseudo- symmetric numerical semigroup*, Boll. Unione Mat. Ital. **9**, 681-696, 2006.

[9] J.C. Rosales, *Numerical semigroups that differ from a Symmetric numerical semigroups in one element*, Algebra Colloq. **15**, 23-32, 2008.

[10] J.C. Rosales and M.B. Branco, *Irreducible numerical semigroups*, Pacific J. Math. **209**, 131-143, 2003.

[11] J.C. Rosales and P.A. García-Sánchez, *Numerical semigroups*, Springer Science & Business Media, 2009.