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# MANIFOLDS WITH NEGATIVE CURVATURE

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ABSTRACT

In this paper, manifolds with negative curvature is discussed, and it is proved, for these manifolds, that the intersection of two compact totally geodesic submanifolds V and W of M does not necessarily occure. This result, is also, established for Kähler manifolds. Finally, the existence of a fixed point is discussed.

## 1- INTRODUCTION

Let M be a complete n dimensional Riemannian manifold, V and W are submanifolds of dimension r and s respectively, and $\tau(0) = p \in V$ to $\tau(\iota) = Q \in W$ striking V and W orthogonally; t represents arc length along. $\tau$. Suppose $x_t$ is a unit vector field that is displaced parallel along $\tau$ and is tangent to V and W at p and q respectively and (if $x_t$ exists) is thus orthogonal to $\tau$. Finally $T_t$ is the unit tangent vector to $\tau$.

We construct a "variation" of the geodesic $\tau$ as follows. We pass a small "ribbon" of surface through $\tau$ that is tangent to $X_t$ at $\tau(t)$ for all t such that $0 \leq t \leq \iota$. This ribbon cuts V and W in two curves. We now pass curve segments on the ribbon tangent to $X_t$ at $\tau(t)$, the curves varying smoothly from V to W. The ribbon is chosen so "thin" that two segments intersect. On each segment we use the directed arc length $\alpha$ from $\tau$ as parameter and we may suppose that $-\varepsilon \leq \alpha \leq \varepsilon$. Each point on the ribbon carries two coordinates $(t, \alpha)$ and we have two systems of coordinate curves t = constant and $\alpha$ = constant (the original geodesic is of course $\alpha = 0$). We have two coordinate vector fields

$T = \dfrac{\partial}{\partial t}$ and $X = \dfrac{\partial}{\partial \alpha}$ defined on the ribbon with $T = T_t$ at $(t, 0)$ and $X = X_t$ at this same point.

We recall some facts and notations of Riemannian geometry. We let g (Y, Z) denote the Riemannian scalar product of two vectors Y and Z; if $(x_1, \ldots, x_n)$ are local coordinates for M, then $g(Y, Z) = \sum_{i,j} g_{ij} Y^i Z^j$.

Levi-Civita connection of a function f with respect to a vector Y, denoted by $\nabla_Y(f)$, is the directional derivative of f in the direction Y. If z is a vector field, the covariant derivative of z with respect to Y is again a vector, written $\nabla_Y z$. If Y is also a vector field, the Lie bracket of Y and Z is given by

$$[Y, Z] = YZ - ZY = \nabla_Y Z - \nabla_Z Y.$$

In particular, if Y and Z are coordinate vectors, then

$$[Y, Z] = 0 = \nabla_Y Z - \nabla_Z Y.$$

Hence in the case of our particular vectors we have

$$\nabla_X T = \nabla_T X$$

where $X = X_t$ is a unit vector field that is displaced parallel along $\tau$ and $T = T_t$ is the unit tangent vector to $\tau$.

Next we have the Ricci operator identity

$$\nabla_Y \nabla_Z - \nabla_Z \nabla_Y = R(Y, Z) + \nabla_{[Y, Z]},$$

where R (Y, Z) is, for each pair (Y, Z), a linear transformation on tangent vectors. R (Y, Z) is constructed from the Riemann curvature tensor and in terms of coordinates the transformation of vectors $U \rightarrow R(Y, Z)\, U$ is given by

$$\sum_i U^i \frac{\partial}{\partial x^i} \rightarrow \sum_i \Big( \sum_{jki} - R^i_{jki} Y^k Z^i U^j \Big) \frac{\partial}{\partial x^i}.$$

R (Y, Z) is skew symmetric; R (Y, Z) = — R (Y, Z). In our case the Ricci identity becomes

$$\nabla_X \nabla_T - \nabla_T \nabla_X = R(X, T).$$

The Riemannian sectional curvature corresponding to the 2- plane $T \wedge X$ is given by

$$\begin{aligned} K(T, X) &= g(R(X, T)\, T, X) \\ &= -g(R(X, T)\, X, T). \end{aligned}$$

Finally, we recall that the scalar product is "covariant constant", i.e.,

$$\frac{\partial}{\partial\alpha} g(Y, Z) = \nabla_X g(Y, Z)$$

$$= g(\nabla_X Y, Z) + g(Y, \nabla_X z).$$

The length of the curve $\alpha$ = constant is given by

$$L(\alpha) = \int_0^{\iota} g(T, T)^{1/2}\, dt.$$

Lemma. The first and second variations of arc length are

$$L'_X(0) = \frac{dL}{d\alpha}\Big|_0 = 0$$

$$L''_X(0) = \frac{d^2L}{d\alpha^2}\Bigg|_0 = g(\nabla_X X, T)_Q - g(\nabla_X X, T)_P - \int_0^{\iota} K(T,X)\, dt.$$

(For a proof see [2]).

2– Real manifolds with negative curvature: A submanifold V of Riemannian M is totally geodesic if any geodesic of M that is tangent to V at a point lies wholly in V. This implies that every geodesic of V (in the naturally induced metric from M) is at the same time a geodesic of M.

Theorem 1. Let M be a complete connected manifold with negative Riemannian sectional curvature. If V and W are compact totally geodesic submanifolds, then V and W have a non-intersection.

Proof. We assume that V and W are any two compact submanifolds. We suppose they intersect. Then there is a largest geodesic $\tau(t)$, say of length $\iota > 0$, from V to W and let P and Q be the points $\tau(0)$ and $\tau(\iota)$ respectively. A variation X for which $L_X''(0) > 0$, hence we arrive at a contradiction and $\tau$ is minimizing.

So far V and W were arbitrary. To evaluate the end term in the second variation we use the fact that V and W are totally geodesic. The variation vector $X_t$ is given. For the construction of the "ribbon", defined in, [5] we can choose geodesics of M through each $X_t$; there is a unit vector $X_0$ tangent to V at P and since V is totally geodesic through $X_\iota$ will lie entirely in W. Thus the curves $\alpha$ = constant will have their endpoints on V and W as required for the variation. But since $X_0$ and $X_\iota$ are tangent vectors to geodesics of M we have $\nabla_X X = 0$ at P and Q. Hence the second variation formula is

$$L''_X(0) = - \int_0^l K(T, X)\, dt > 0,$$

and the proof is complete.

3– Kähler manifolds with negative curvature: A Kähler manifold M is a special type of Riemannian manifold whose underlying space is a complex manifold. There is a linear transformation J on each tangent space that sends any vector Y into a vector JY orthogonal to Y (J represents multiplication by $(-1)^{2/1}$). J has the properties $J^2 = -$ identity and $g(JY, JZ) = g(Y, Z)$ for all vectors Y and Z (the last property states that g is a "Hermitian" metric). From J we construct the Kähler exterior 2- form $\omega$, defined by

$$\omega(Y, Z) = g(JY, Z).$$

$\omega$ is exterior because $\omega(Y, Z) = -\omega(Z, Y)$. All that has been said so far holds for any Hermitian manifold. The further condition defining a Kähler manifold can be stated as requiring that $\omega$ be covariant constant, $\nabla_u \omega = 0$ for all vectors u; i.e., for any vector fields Y and Z we have

$$\nabla_u \omega(Y, Z) = \omega(\nabla_u Y, Z) + \omega(Y, \nabla_u Z).$$

Since g is also covariant constant we conclude that J is also, i.e., we have the operator equation

$$\nabla_u \circ J = J \circ \nabla_u, \qquad (*)$$

for any vector u.

A linear subspace V of the tangent space to a complex manifold at a point is said to be complex if it is invariant under J, $J : V \to V$. A submanifold is complex analytic if its tangent space at each point is complex.

Theorem 2. Let $M_n$ be a complete, connected Kähler manifold with negative sectional curvature. If V and W are compact complex analytic submanifolds, then V and W we have nonintersect.

Proof. The proof is again by contradiction starting exactly as in Theorem 1. We again arrive at a variation vector $X_t$, parallel displaced along $\tau$ and tangent to V and W at P and Q respectively. Now, however, we have additional information. Since V and W are complex analytic the vector field $J(X_t)$ is tangent to V and W at P and Q respectively. Further, from (*) we have

$$\nabla_T J(X_t) = J \nabla_t (x_t) = 0.$$

since $X_t$ is parallel displaced. Thus $J(X_t)$ is also parallel displaced and gives the same type of variation vector as $X_t$. We claim, that the second variation corresponding to at least one of the fields $X_t$ or $J X_t$ is strictly positive again giving a contradiction.

To prove our claim we suppose that

$$L''_X(0) = g(\nabla_X X, T)_Q - g(\nabla_X X, T)_P - \int_0^t K(T, X)\, dt \leq 0.$$

By the hypothesis of negative curvature we conclude that

$$g(\nabla_X X, T)_Q - g(\nabla_X X, T)_P < 0.$$

We will be finished if we can show

$$g(\nabla_{JX} JX, T)_Q - g(\nabla_{JX} JX, T)_P > 0.$$

Since every second fundamental form of a complex analytic submanifold of a Kähler manifold is skew hermitian; i.e.,

$$g(\nabla_{JX} JX, T)_P = -g(\nabla_X X, T)_P \qquad \text{for } V,$$

$$g(\nabla_{JX} JX, T)_Q = -g(\nabla_X X, T)_Q \qquad \text{for } W.$$

The proof of this is simple and we include it here for completeness.

Let C be a complex analytic curve (real dimension 2) on V tangent to $X_0$ and $JX_0$ at P. Then $X_0$ can be extended to a tangent vector field X on C and of course JX is an extension of $JX_0$. Since X and JX are tangent vector fields to C the lie bracket $[JX, X]$ is again a vector field tangent to C, and thus orthogonal to T at P. Using $[JX, X] = \nabla_{JX} X - \nabla_X JX$, (4) and $J^2 = -$ identity we have at P,

$$\begin{aligned} g(\nabla_{JX} JX, T) &= g(J \nabla_{JX} X, T) \\ &= g(J([JX, X] + \nabla_X JX), T) \\ &= g(J [JX, X], T) - g(\nabla_X X, T). \end{aligned}$$

Since $[JX, X]$ is tangent to C, so is $J(JX, X]$ and so the first term vanishes and the result follows.

4– Correspondences: A (holomorphic) correspondence of a complex manifold $C_n$ with itself is a complex analytic n-dimensional submanifold of $C_n \times C_n$.

A holomorphic map $f : C_n \to C_n$ gives rise to a correspondence, the graph $G(f)$ of f; $G(f) = \{(p, f(p)) : p \in C_n\}$. G(f) is of course a special type of correspondence since f is single valued. Let $\triangle = \{(p, p) : p \in C_n\}$ be the diagonal of $C_n \times C_n$. A correspondence will be said to have a fixed point if G (f) and $\triangle$ are intersects the diagonal.

Proposition 1. Every (holomorphic) correspondence of a connected compact Kähler manifold $C_n$ with negative curvature has a non fixed point. Proof. The holomorphic is a complex analytic submanifold $V_n$ of $C_n \times C_n$. The same is true for the diagonal $\triangle$. We need only show that $V_n$ and the diagonal $\triangle$ is not intersect, and this follows from Theorem 2. The proof is complete.

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