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ON RATES OF CONVERGENCE AND DIVERGENCE OF THE CAUCHY PRODUCT OF SERIES

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ABSTRACT

At first, we show that the Cauchy product of two series with positive real terms does not converge faster than given series. Later, we state the theorems on rates of divergence of Cauchy product of series.

1– *Basic facts and definitions :* Throughout the paper all series are considered with positive real terms. Let $\sum_{n=o}^{\infty} a_n$ and $\sum_{n=o}^{\infty} b_n$ be two series and let p be a positive integer. If

$$c_p = \sum_{m+n=p} a_m b_n$$

then $\sum_{p=o}^{\infty} c_p$ is called the Cauchy product of $\sum_{n=o}^{\infty} a_n$ and $\sum_{n=o}^{\infty} b_n$.

Now we give two known theorems on Cauchy product of series.

Theorem A : If $\sum_{n=o}^{\infty} a_n = A$ and $\sum_{n=o}^{\infty} b_n = B$ are absolutely convergent series then the Cauchy product $\sum_{n=o}^{\infty} c_n = C$ of these series is absolutely convergent and $C = A.B$. [2]

Theorem B : Let (a_n) and (b_n) be two sequences of real numbers with $a_n \to 0$ $(n \to \infty)$ and $\sum_{n=o}^{\infty} |b_n| < \infty$. If Cauchy product of

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$\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is $\sum_{n=0}^{\infty} c_n$ then $c_p = o(1)$ $(p \to \infty)$. [2]

Now, we state the definitions on rates of convergence and divergence of the series with positive real terms are given in [1]. In this definitions, term-by-term quotients are considered.

Definition 1: Let $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$ be two convergent series of positive real numbers and let the sequences of partial sums of these series be (s_k) and (s_k'), respectively. We take $r_k = s_k - A$, $r_k' = s_k' - B$.If

$$\lim_{k\to\infty} r_k'/r_k = 0 \text{ (or } +\infty)$$

then $\sum_{n=0}^{\infty} b_n$ converges faster (or slower) than $\sum_{n=0}^{\infty} a_n$.

If $0 < \liminf_{k\to\infty} r'_k/r_k \leq \limsup_{k\to\infty} r'_k/r_k < +\infty$

then $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge at the same rate.

Definition 2: Let $\sum_{n=0}^{\infty} c_n$ and $\sum_{n=0}^{\infty} d_n$ be two divergent series of positive real numbers and let the sequences of partial sums of these series be (s_k) and (s'_k), respectively. If

$$\lim_{k\to\infty} s'_k/s_k = +\infty \text{ (or } 0)$$

then $\sum_{n=0}^{\infty} d_n$ diverges faster (or slower) than $\sum_{n=0}^{\infty} c_n$.

If

$$0 < \liminf_{k\to\infty} s'_k/s_k \leq \underset{k\to\infty}{s'_k/s_k} < +\infty$$

then $\sum_{n=0}^{\infty} c_n$ and $\sum_{n=0}^{\infty} d_n$ diverge at the same rate.

2– We have the following theorem on rate of convergence of Cauchy product of series.

Theorem 1: Let $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$ be two convergent series of positive real numbers. Then the Cauchy product $\sum_{n=0}^{\infty} c_n$ of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ does not converge faster than $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

Proof: From Theorem A, we have $\sum_{n=0}^{\infty} c_n = C = A.B$. Let (s_n), (s'_n), (s''_n) be the sequences of partial sums of $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, $\sum_{n=0}^{\infty} c_n$, respectively. We take $r_n = A—s_n$, $r'_n = B—s'_n$, $r''_n = C—s''_n$. For all positive integers n, we have

$$r''_n / r'_n = (c_{n+1} + c_{n+2} + \ldots) / (b_{n+1} + b_{n+2} + \ldots)$$
$$> \frac{a_0(b_{n+1}+b_{n+2}+\ldots)+a_1(b_{n+1}+b_{n+2}+\ldots)+\ldots}{b_{n+1}+b_{n+2}+\ldots} = a_0+a_1+\ldots=A$$

Therefore r''_n / r'_n does not tend to zero, because of $\sum_{n=0}^{\infty} a_n$ is a convergent series of positive numbers and $0 < A < +\infty$. Hence $\sum_{n=0}^{\infty} c_n$ does not converge faster than $\sum_{n=0}^{\infty} b_n$.

At the same way r''_n / r_n does not tend to zero, i.e. $\sum_{n=0}^{\infty} c_n$ does not converge faster than $\sum_{n=0}^{\infty} a_n$.

From the proof of Theorem 1, we see that the Cauchy product and given series may converge at the same rate or the Cauchy product converges slower than given series.

Now we state the theorems on rate of divergence of Cauchy product of series which are the analogues of Theorems given in [3].

Theorem 2: Let (a_n) and (b_n) be two sequences of positi ve numbers with $\lim_{n=\infty} a_n = 0$ and $\sum_{k=0}^{\infty} b_k < \infty$. And let $\sum_{n=0}^{\infty} c_n$ be Cauchy product of $\sum_{n\to 0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Suppose that for each increasing sequence (k_j) of positive integers, $\Sigma\, a_{k_j}$ and $\Sigma\, c_{k_j}$ both converge or both diverge. Then $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} c_k$ either both converge or both diverge at the same rate.

Proof: From Theorem B, $c_k = o(1)$ $(k \to \infty)$ holds. If every subseries of $\sum_{k=0}^{\infty} a_k$ (and hence of $\sum_{k=0}^{\infty} c_k$) converges, there is nothing to prove. So suppose $\Sigma\, a_k$ has some convergent and some divergent subseries. If the conclusion of Theorem 2 does not hold, then we may assume

$$(1) \qquad \limsup_{n\to\infty} t_n/s_n = +\infty.$$

Where, $t_n = \sum_{k=1}^{n} c_k$ and $s_n = \sum_{k=1}^{n} a_k$. Consider the set S of positive integers defined by $S = \{k: c_k \geqslant a_k\}$. Let $k_1, k_2, \ldots$ denote the elements of S listed in increasing order. By (1), it follows that Σc_{k_j} and Σa_{k_j} diverge and

$$\limsup_{n\to\infty} \tilde{t}_{k_n}/\tilde{s}_{k_n} = +\infty$$

Where, $\tilde{t}_{k_n} = \sum_{j=1}^{n} c_{k_j}$ and $\tilde{s}_{k_n} = \sum_{j=1}^{n} a_{k_j}$.

We now reassociate the terms of Σc_{k_j} as follows. Let

$$\alpha_1 = c_{k_1} + c_{k_2} + \ldots + c_{k_{n_1}}$$

Where, $n_1 = \min \{n : \sum_{j=1}^{n} c_{k_j} \geqslant 1\}$. Having defined $\alpha_1, \alpha_2, \ldots, \alpha_m$ and associated positive integers $n_1 < \ldots < n_m$, let

$$\alpha_{m+1} = c_{k_{n_{m+1}}} + + \ldots + c_{k_{n_{m+1}}}$$

Where $n_{m+1} = \min \{n: \sum_{j=n_m+1}^{n} c_{k_j} \geqslant 1\}$. We then define (β_m) by

$$\beta_m = \sum_{j=n_{m-1}+1}^{n_m} a_{k_j}, \; m = 1, 2, 3, \ldots \text{ (taking } n_0 = 0).$$

It then follows that

$$(2) \quad \limsup_{n\to\infty} \sum_{m=1}^{n} \alpha_m \Big/ \sum_{m=1}^{n} \beta_m = +\infty$$

and since $c_k \to 0$ $(k \to \infty)$,

$$(3) \quad \lim_{m\to\infty} \alpha_m = 1.$$

Now from (2) we must have $\liminf_{m\to\infty} \beta_m/\alpha_m = 0$. Thus we may select an increasing sequence (m_j) of positive integers with

$$(4) \quad \beta_{m_j}/\alpha_{m_j} \leq 1/j^2, \; j=1,2,\ldots.$$

By (3) and (4) we see that $\sum_{j=1}^{\infty} \alpha_{m_j}$ diverges while $\sum_{j=1}^{\infty} \beta_{m_j}$ converges. Seperating the α_{m_j}'s and β_{m_j}'s into their component c_{k_j}'s and a_{k_j}'s gives a subseries of $\sum_{k=0}^{\infty} c_k$ that diverges while the corresponding subseries of $\Sigma\, a_k$ converges. This contradiction shows that (1) cannot hold, this completes the proof.

We have the following results as immediate corollaries of Theorem 2.

Corollary 1: Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be two series satisfying the hypotheses of Theorem 2 and let $\sum_{k=0}^{\infty} c_k$ be the Cauchy product of $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$. Let (k_j) be an increasing sequence of positive integers. Then $\Sigma\, a_{k_j}$ and $\Sigma\, c_{k_j}$ either both converge or both diverge at the same rate.

Corollary 2: Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be two series satisfying the hypotheses of Theorem 2 and let $\sum_{k=0}^{\infty} c_k$ be the Cauchy product of $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$. Let π be any permutation of the positive integers. Then $\Sigma\, a_{\pi(k)}$ and $\Sigma\, c_{\pi(k)}$ either both converge or both diverge at the same rate.

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