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ON 6- FIGURES IN MOUFANG PROJECTIVE PLANES

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ABSTRACT

In this paper, the concept of cross-ratio is extended to the whole Moufang plane and some properties of 6-figures in the π Moufang plane is examined. Essentially the geometric properties of the π which is equivalent to the existence of a square root of an element of an alternative division ring R is determined.

INTRODUCTION

Let Π be a Moufang projective plane. It is well known that π determines a unique alternative ring R. One of the main problems in projective geometry is to find geometric properties of π which are equivalent to certain algebraic properties of the alternative ring R. For instance, Π is Desarguesian if and only if R is associative. In this paper, we extend some of the properties of the 6-figures, which have been given by Cater [1] for Desarguesian planes, to the Moufang planes. Namely, we determined the geometric properties of Π equivalent to the existence of a square root of an element of R. And if (A,B:C,D) denotes the cross-ratio of distinct collinear points A,B,C,D in Π, we construct points J and N such that $(A,B:C,J) = (A,B:C,D)^2$ and $(A,B:C,N) = (A,B:C,D)^3$. In fact, these are investigated in [2]. But some mappings which is obtained by using the identity

(1) $$x^{-1}(y(xz)) = (x^{-1}\ yx)z$$

asserted in [5] for Cayley-Dickson algebras are used. However [3] demonstrated here that (1) does not valid for Cayley-Dickson algebras. But also the existence of 4-point transitivity is shown by using of some new mappings. Essentially, in this paper, [2] is rearranged and modified under the light of [3].

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Throughout the paper we use the terminology in [1] and [3]: A 6–figure is a sequence of 6 distinct points (ABC, A′B′C′) such that ABC is a triangle, and $A'\varepsilon BC$, $B'\varepsilon CA$, $C'\varepsilon AB$. The points A,B,C,A′,B′,C′ are called vertices of this 6–figure. A 6–figure (ABC, A′B′C′) is said to be equivalent to any 6–figure (DEF,D′E′F′) if there exist a projective collineation of Π which maps A,B,C,A′,B′,C′ to D,E,F,D′,E′,F′ respectively; in symbols $ABCA'B'C' \barwedge DEFD'E'F'$.

(ABC, A′B′C′) is called a menelaus 6–figure if A′,B′,C′ are collinear; and (ABC,A′B′C′) is called a ceva 6–figure if the lines AA′,BB′,CC′ are concurrent.

Throughout the paper, we assume that R is an alternative division ring with center K of arbitrary characteristic and that Π is the Moufang plane coordinatized by R, [4].

It is easy to see that the mappings

$$I_1:(x,y) \to (x^{-1}, yx^{-1}),\ x \neq 0,\ (0,y) \leftrightarrow (y),\ (\infty) \to (\infty)$$

$$[m,k] \to [k,m],\ [k] \to [k^{-1}],\ k \neq 0,\ [0] \leftrightarrow [\infty]$$

and

$$I_2:(x,y) \to (xy^{-1},y^{-1}),\ y \neq 0,\ (x,0) \leftrightarrow (x^{-1}),\ x \neq 0,\ (0,0) \leftrightarrow (\infty),\ (0) \to (0)$$

$$[m,k] \to [-k^{-1}m,k^{-1}],\ k \neq 0,\ [m,0] \leftrightarrow [m^{-1}],m \neq 0,\ [0,0] \leftrightarrow [\infty],\ [0] \to [0]$$

are collineations of Π.

Let A=(0), B=(∞), C=(0,0), A′=(0,1), B′=(–1,0), C′=(–m) for some $m\varepsilon R$. Here, (–1,0) (∞). (0,1) (0) = (–1,1) and

$$I_2I_1:(0),(\infty),(0,0),(-1,1) \to (\infty),\ (0,0),\ (0),\ (1,-1).$$

According to the Theorem 1 in [3], the mapping g which maps (0),(∞), (0 0),(1,–1), to (0),(∞),(0,0),(–1,m) is a composition of the collineations F_c and $S\alpha,\beta$ where $c\varepsilon R$, $\alpha,\beta\varepsilon K$. We already known that such a collineation, maps (x,y) to (x′,yd). Now, since g maps (1,–1) to (–1,m) than (–1) d = m and so $-m^{-1} = d^{-1}$.

Therefore $(0,-m^{-1})$ maps to $(0,-m^{-1}\, d) = (0,1)$. Thus

$$f = gI_2I_1:(0),(\infty),(0,0)(0,1),(-1,0),(-m) \to (\infty),(0,0),(0),(-1,0),(-m),(0,1).$$

Consequently

$$ABCA'B'C' \barwedge BCAB'C'A' \barwedge CABC'A'B'$$

is shown by using f and f^2.

Furthermore, any 6–figure (DEF,D′E′F′) is equivalent to ((0)(∞)(0,0),(0,1)(–1,0)(–m)) for some mεR; since there exist by Theorem 1 in [3], a collineation mapping D,E,D′,E′ 4–point on (0), (∞), (0,1), (–1,0). Thus (ABC,A′B′C′), (BCA, B′C′A′), (CAB,C′A′B′) will be regarded as the same 6–figure μ, and likewise (ACB,A′C′B′),(CBA,C′B′A′), (BAC,B′A′C′) as the same 6–figure λ. μ and λ are called opposite 6–figures of each other; in symbols, $\lambda = \mu^{-1}$ and $\mu = \lambda^{-1}$.

Let Π be a Moufang plane satisfying the Fano's Axion. It follows, (see [7]) that there exist unique points A″ε BC,B″ε CA, C″ εAB such that H(AB,C′C″), H(BC,A′A″),H(CA,B′B″). The 6–figure (ABC,A″B″C″) is called the conjugate of μ, in symbol - μ. Likewise μ is the conjugate of - μ.

Let $C^d\varepsilon$AB be the point such that C,C^d and AA′. BB′ are collinear. Let $A^d\varepsilon$BC and $B^d\varepsilon$CA be the points such that A,A^d and BB′.CC′ are collinear and B,B^d and AA′.CC′ are collinear. The 6–figure (ACB, A^d C^dB^d) is called the first descendant of μ, written μ^d. μ is claled a first ancestor of μ^d.

Let A^c = BC . B′C′, B^c = CA.C′A′, C^c = AB . A′B′. The figure (ACB,$A^cC^cB^c$) is called the first codescendant of μ, written μ^c . μ is called a first coancestor of μ^c.

Figure 1 represents the 6–figure μ_1 = ((0) (∞) (0,0), (0,1) (–1,0) (–m)) and $\mu_1{}^c$ and construction of B^{cc}. In figure 2 μ_1, $\mu_1{}^d$ are drawn and B^{dc} is constructed. In figure 3 the points A''^d, B''^d, C''^d, B''^c are drawn.

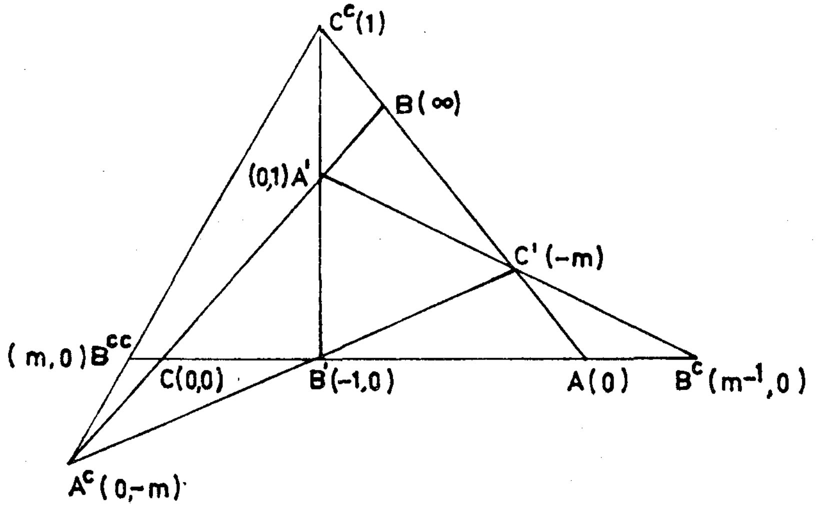

Figure 1

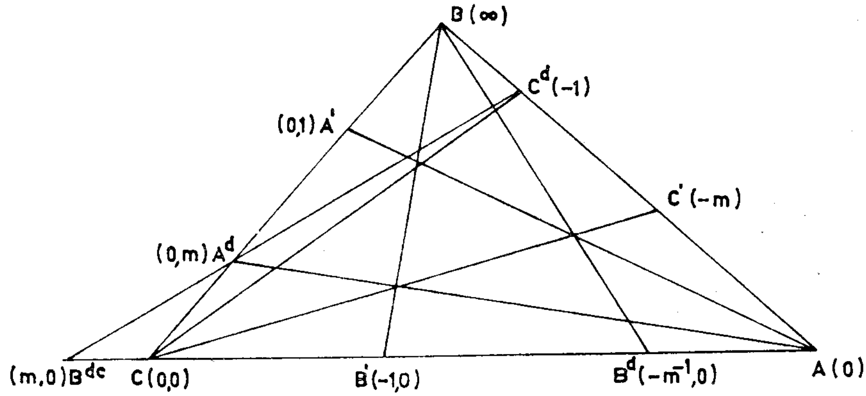

Figure 3

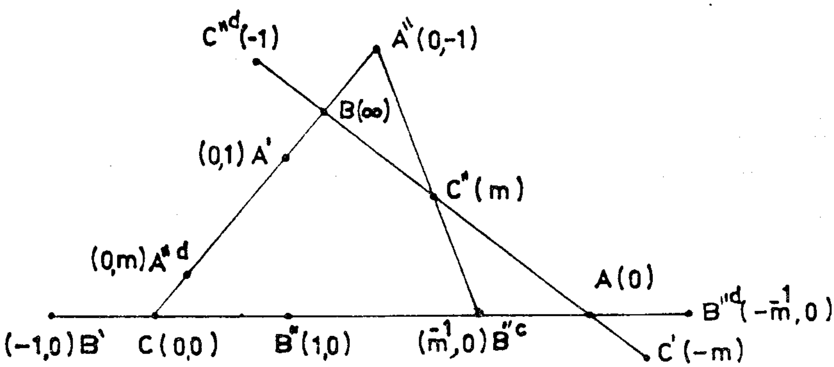

Figure 2

SOME PROPERTIES OF 6–FIGURES IN MOUFANG PLANES

The classical definition of the cross-ratio for Desarguesian planes is not available for the Moufang planes. Ferrar [5] gives the following algebraic definition of the cross-ratio for the points on the line [0,0].

$$(A,B:C,D) = (a,b:c,d) = \{((a-d)^{-1}(b-d))((b-c)^{-1}(a-c))\}$$

where A = (a,0), B = (b,0), C = (c,0), D = (d,0) and $\{x\}$ denotes the conjugancy class of x in the alternative ring R. We extend this definition to whole plane as follows:

(i) If $A = (a,a_1)$, $B = (b,b_1)$, $C = (c,c_1)$, $D = (d,d_1)$ are on a line of type $[m,k]$ let $(A,B:C,D) = (a,b:c,d)$. For this case one of A,B,C,D is the ideal point (m), use $\infty \notin R$ instead of the corresponding component in $(a,b:c,d)$. (Notice the perspectivity from $[m,k]$ to $[0,0]$ with center (∞)).

(ii) If A,B,C,D are on a line of type $[k]$ that is, $A = (k,a)$, $B = (k,b)$, $C = (k,c)$, $D = (k,d)$ let $(A,B:C,D) = (a,b:c,d)$. In the case one of A,B,C,D is (∞) use again ∞ in $(a,b:c,d)$. (Notice that the perspectivity from $[k]$ to $[0,0]$ with center (-1) maps $(k,x) \to (k+x,0)$; and taking $k+x$ instead of x does not change the cross ratio.).

(iii) If the points are on $[\infty]$, that is $A = (a)$, $B = (b)$, $C = (c)$, $D = (d)$, let $(A,B:C,D) = (a,b:c,d)$. (Notice that the perspectivity from $[\infty]$ to $[0,0]$ with center $(0,-1)$ maps $(m) \to (m^{-1},0)$, $(0) \to (0)$, $(\infty) \to (0,0)$, and $(a,b:c,d) = (a^{-1}, b^{-1}: c^{-1}, d^{-1})$.).

LEMMA 1. In a Moufang plane a perspectivity preserve the cross ratio.

Proof: Let $A,B,C,D \varepsilon l$, $\varnothing_M$ and $\varnothing_N$ be perpectivities from any line l to $[0,0]$ which have center M and N, respectively. Where $\varnothing_M$ is the perpectivity in defination of generalized cross - ratio, and $N \neq M$, $N \varepsilon l$, $N \varepsilon [0,0]$. In addition let $\varnothing_M$: $A,B,C,D \to A_1,B_1,C_1,D_1$ and $\varnothing_N$: $A,B,C,D \to A_0,B_0,C_0,D_0$.

To prove, it is sufficient to show that $\varnothing_N$ preserves the cross-ratio. According to the definition $(A,B:C,D) = (A_1B_1:C_1D_1)$. Thus

$$\varnothing = \varnothing_M \varnothing_N^{-1}: A_0,B_0,C_0,D_0 \to A_1,B_1,C_1,D_1$$

Therefore $\varnothing$ is a projectivity of $[0,0]$.

Since $\varnothing$ is a projectivity of $[0,0]$ which preserves the cross-ratio (See [5], Theorem 3,7) $(A_1,B_1:C_1,D_1)$ is equal to $(A_0,B_0:C_0,D_0)$. Consequently $(A,B:C,D) = (A_0,B_0:C_0,D_0)$, that is, $\varnothing_N$ preserves the cross-ratio.

In what follows we use the fact that distinct collinear poinst A,B,C, D in Π are in harmonic position if and only if $(A,B:C,D) = -1$.

LEMMA 2. If $\mu_1 = (ABC,A'B'C') = ((0)\ (\infty)\ (0,0),\ (0,1)\ (-1,0)\ (-m))$ then

$$(A,B:C',C^c) = (B,C:A',A^c) = (C,A:B',B^c) = \{-m\}.$$

Proof is trivial.

Conjugacy class of $-(A,B:C',C^c) = -(C,A:B',B^c)$ is called the ratio of the 6–figure $\mu = (ABC,A'B'C')$, denoted by $r(\mu)$.

THEOREM 1:

(i) μ is a menelaus 6–figure if and only if $r(\mu) = -1$

(ii) μ is a ceva 6–figure if and only if $r(\mu) = 1$

Proof: (i) It suffices to assume that μ is μ_1 because cross-ratio is preserved by projective collineations. Thus $\mu_1 = ((0)\ (\infty)\ (0,0),\ (0,1)\ (-1,0)\ (1))$ and $r(\mu_1) = -1$.

Conversely, if $r(\mu_1) = -1$; than $\mu_1 = ((0)\ (\infty)\ (0,0),\ (0,1)\ (-1,0)\ (1))$ and the points of (0,1), (–1,0), (1) are collinear.

(ii) If μ_1 is a ceva 6–figure, than $\mu_1 = ((0)\ (\infty)\ (0,0),\ (0,1)\ (-1,0)\ (-1))$ and $r(\mu_1) = 1$.

Conversely, if $r(\mu_1) = -1$, μ_1 as above thus, the lines of (0) (0,1), (∞) (–1,0), (0,0) (–1) is concurent.

THEOREM 2: For any 6–figure μ we have

(i) $r(\mu^{-1}) = (r(\mu))^{-1}$

(ii) $r(-\mu) = -r(\mu)$

(iii) $r(\mu^d) = (r(\mu))^2$

(iv) $r(\mu^c) = -(r(\mu))^2$

Proof: It is sufficient to assume that μ is μ_1 in figures 1,2,3 since cross ratio is preserved by projective collineations. With a simple calculation we have

(i) $r(\mu_1^{-1}) = -(\infty,0:-1,m^{-1}) = \{m^{-1}\} = (r(\mu_1))^{-1}$;

(ii) $r(-\mu_1) = -(0,\infty:1,m^{-1}) = \{-m\} = -(r(\mu_1))$;

(iii) $r(\mu_1^d) = -(\infty,0:-m^{-1},m) = \{m^2\} = (r(\mu_1))^2$

(iv) $r(\mu_1^c) = -(\infty,0:m^{-1},m) = \{-m^2\} = -(r(\mu_1))^2$;

It follows immediately, from theorem 1 and 2 that conjugate of a menelaus 6–figure is ceva 6–figure and vice versa. In fact this has been shown in [6].

THEOREM 3. Let $m \varepsilon R$, $m \neq 0$. Then the equation $x^2 = m$ (or $x^2 = -m$) has a solution in R if and only if any 6–figure μ with ratio $\{m\}$ has a first ancestor (coancestor) in Π.

Proof: Without loss of generality we take $\mu = \mu_1$. Let $\lambda = ((0)(0,0)(\infty), DEF)$ be a first ancestor of μ_1. Then we have $\{q^2\} = \{m\}$ by Theorem 1, and there exists an $x \varepsilon R$ satisfying $x^2 = m$.

Conversely suppose that there exist $q \varepsilon R$ such that $q^2 = m$. In this case it is easily computed that $\lambda_1 = ((0)(0,0)(\infty),(0,-q)(q)(q^{-1},0))$ is a first ancestor of μ_1.

Now consider the equation $x^2 = -m$. Let $\lambda_1 = ((0)(0,0)(\infty), STU)$ be a first coancestor of μ_1 and let $r(\lambda_1) = \{q\}$. Then we have $\{m\} = -\{q^2\}$ by Theorem 1, and there exists an $x \varepsilon R$ satisfying $x^2 = -m$.

Suppose $q \varepsilon R$ and $q^2 = -m$. Then one can easily show that $\lambda_1 = ((0)(0,0)(\infty),(0,q)(q)(-q^{-1},0))$ is a first coancestor of μ_1.

Clearly one can easily observe from the figures that first descandant of μ and $-\mu$ are same, and that first codescandant of μ and $-\mu$ are same. Namely $\mu^d = (-\mu)^d$ and $\mu^c = (-\mu)^c$.

Finally, it is worth to note that the construction which gives an algorithm for "squaring" a cross-ratio of points in a Desarguesian plane ([1]) can be also extended to a Moufang plane as follows:

Let A,B,C,D be any collinear points in Π. Choose any point E not on AB, and choose points $F \varepsilon AE$, $G \varepsilon BE$ such that F,G and C are collinear points.

Let I = AE . DG, H = BE . DF and J = AB . HI . For this project A,B,F,G to (0),(0,0),(–1),(0,1) respectively. Thus $(A,B:C,D) = \{m\}$ and $(A,B:C,J) = \{m^2\}$ by Lemma 1, and consequently $(A,B:C,J) = (A,B:C,D)^2$.

Furthermore, if K = AE . CH, M = AB . KG and G′ = BE . MF, I′ = AE . G′D, N = AB . HI′ then $(A,B:C,N) = (A,B,:C,D)^3$.

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