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SOME PROPERTIES WHICH A RING OF HOLOMORPHIC FUNCTIONS ON A NON-EMPTY SUBSET OF AN OPEN RIEMANN SURFACE MIGHT HAVE

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ABSTRACT

Su (1972) proved that for any two subsets X, Y of C, the complex plane, X and Y are conformally homeomorphic if there is an isomorphism between H(X) and H(Y) which is the identity on constant functions. Minda (1976) extended the method to the rings of holomorphic functions on any subsets of open Riemann surfaces. Royden (1963) listed some properties which a ring of functions might have. In this paper, using the results of Su, Minda, and Royden we present some properties of the subring $R\varphi$.

Introduction

In this paper R and S will denote open Riemann surfaces and X,Y will be non-empty subsets of R and S, respectively. A function $\varphi: X \to S$ is said to be analytic if for each point $p \in X$ there is an open neighborhood U_p of p and an **analytic** function $\psi_p: U_p \to S$ such that ψ_p and φ coincide on $U_p \cap X$. This is equivalent to assuming that there is a single open set $U \supset X$ and an analytic function $\psi: U \to S$ such that $\psi|X = \varphi$, (Minda, 1973); in this reference it is assumed that R=S=C, but the method readily extends to the present situation. Let A(X,Y) denote the set of all analytic functions $\varphi: X \to S$ with $\varphi(X) \subset Y$. For Y=S=C, a function in A(X,C) is called **holomorphic** and we write A(X,C) = H(X). Thus H(X) is the set of all holomorphic functions on X which is non-empty subset of an open Riemann surface R. It is well known that H(X) is an integral domain under pointwise addition and multiplication. In fact, H(X) is an algebra over both the complex numbers C and the real numbers $\mathcal{R}$.

It is known that if R and S are open Riemann surfaces and X,Y non-empty subsets of R,S respectively, and if Φ is a C-algebra homo-

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morphism of H(Y) into H(X) *mapping each constant function onto* itself, then there is a unique analytic mapping $\varphi \varepsilon A(X,Y)$ such that $\Phi(g) = go\varphi$ for $g \varepsilon H(Y)$, Minda (1976). Also if Φ is an isomorphism of H(Y) into H(X), then φ is a one-to-one mapping of X into Y.Thus a subring R* of H(X) is a homomorphic image of a ring H(Y) under a C-algebra homomorphism if and only if $R^* = R_\varphi = \{go\varphi: g \varepsilon H(Y), \varphi \varepsilon A(X,Y)\}$. More over, if φ is a one-to-one analytic mapping of X onto Y and Φ maps H(Y) into H(X) in such a way that $\Phi(g) = go\varphi$, $g \varepsilon H(Y)$, then $\Phi(H(Y)) = H(X)$.

In this paper we are concerned with the proper subrings R* of H(X) which are C-isomorphic images of H(Y), the algebra of all holomorphic functions on a non-empty subset Y of an open. Riemann surface S. Royden (1963) listed the following properties which a ring of functions might have. He and others have shown that if G is an open set in the plane or on open Reimann surface, then H(G) has these properties.

(α) If $f \varepsilon H(G)$ and if f is never zero, then f has an multiplicative inverse in H(G).

(β) If $f_1,\dots,f_n$ are elements of H(G) with no common zeros, then there are elements $e_1,\dots,e_n$ in H(G) such that $e_1f_1 + \dots + e_nf_n = 1$.

(Υ) If $f \varepsilon H(G)$ and f is not identically zero, there are a finite number of functions $f_1,\dots,f_n$ in H(G) which separate the zeros of f.

According to Minda (1976), in case that Y is a non-empty subset of an open Riemann surface S,H(Y) the algebra of holomorphic functions on Y has the following similar properties:

(α^*) If $f \varepsilon H(Y)$ and $f(z) \neq 0$ for every $z \varepsilon Y$, then $\frac{1}{f} \varepsilon H(Y)$.

(β^*) If $f_1,\dots,f_n$ *belong to* H(Y) and have no common zero, then there are functions $e_1,\dots,e_n$ in H(Y) such that $e_1f_1 + \dots + e_nf_n = 1$,

(Υ^*) If $f \varepsilon H(Y)$ and $f \not\equiv 0$, then there is a set $\{f_1,\dots,f_n\}$ contained in H(Y) such that $x \neq y$ and $f(x) = f(y) = 0$, then there is a function $f_i, i=1,\dots,n$, such that $f_i(x) \neq f_i(y)$.

In the next section we shall establish whether the proper subrings of H(X) which are C-isomorphic images of H(Y) will have these properties. But before passing to the next section let us state the necessary additional properties of proper subrings of H(X).

Suppose X and Y non-empty subsets of open Reimann surfaces R and S respectively. We define a mapping of H(Y) into H(X) by $\Phi(g) = go\varphi$ for $g \varepsilon H(Y)$. $go\varphi$ is holomorphic on X and Φ is a C-algebra homomorphism. The image of Φ, $R_\varphi = \Phi(H(Y))$ is a subring of H(X). R_φ contains the constant functions, denoted by C, since $C \subseteq H(Y)$ and $\Phi(\lambda) = \lambda$ for $\lambda \varepsilon C$.

Now, if $\Phi: H(Y) \rightarrow H(X)$ is a ring homomorphism defined by $\Phi(g) = go\varphi$ for $g \varepsilon H(Y)$, $\varphi \varepsilon A(X,Y)$ and if $R_\varphi = \Phi(H(Y))$, then the following three conditions are equivalent:

(a) R_φ properlty contains the constant functions.

(b) φ is not a constant function.

(c) H(Y) is isomorphic to R_φ.

It is clear that these are the relations between Φ, φ, and R_φ. Thus a subring R^* of H(X) is isomorphic to H(Y) under a C-algebra isomorphism, if only if $R^* = \{go\varphi: g \varepsilon H(Y), \varphi \varepsilon A(X,Y)\}$ and R^* properly contains C the constant functions on X.

More over, if φ is a one-to-one analytic mapping of X into Y, λ is a non-constant analytic mapping of X into Y but not one-to-one, $\Phi(g) = go\varphi$ and $\Lambda(g) = go\lambda$ for $g \varepsilon H(Y)$, $R_\varphi = (H(Y))$, $R_\lambda = \Lambda(H(Y))$, then R_φ and R_λ are isomorphic but $R_\varphi \neq R_\lambda$.

The Function φ Determines Whether R_φ Will Have Properties (α^*), (β^*), And (Υ^*).

Theorem 1. If φ is an analytic mapping of X onto Y, then R_φ has property (β^*).

Proof. Let $f_1,\ldots,f_n$ belong to R_φ and have no common zero. $f_i = \Phi(h_i)$, $i=1,\ldots,n$, where $h_i \varepsilon H(Y)$. Suppose $h_i(a) = 0$ for $a \varepsilon Y$, $i=1,\ldots,n$. Since φ maps X onto Y, there is $z \varepsilon X$ such that $\varphi(z) = a$. Then $0 = h_i(a) = h_i(\varphi(z)) = \Phi(h_i(z)) = f_i(z)$ for $i=1,\ldots,n$. This is a contradiction, thus $h_1,\ldots,h_n$ can have no common zero, H(Y) has property (β^*) implies there are $e_1,\ldots,e_n$ in H(Y) such that $e_1h_1+\ldots+e_nh_n = 1$.

$$1 = \Phi(e_1h_1 + \ldots + e_nh_n) = \Phi(e_1h_1) + \ldots + \Phi(e_nh_n)$$
$$= \Phi(e_1)\,\Phi(h_1) + \ldots + \Phi(e_n)\,\Phi(h_n) = \Phi(e_1)\,f_1 + \ldots + \Phi(e_n)\,f_n$$
which implies there are functions $\Phi(e_i)$, $i=1,\ldots,n$, *in* R_φ *such that* $\Phi(e_1)\,f_1 + \ldots + \Phi(e_n)\,f_n = 1$.

If a set of functions has property (β^*), then it has property (α^*) because when $n=1$, (β^*) is (α^*). If φ is an onto mapping, R_φ has properties (α^*) and (β^*).

Theorem 2. If R_φ has property (α^*) and R_φ properly contains the constant functions, then φ maps X onto Y.

Proof. Let $a \varepsilon Y$. Since H(S) is an algebra of all holomprhic functions on the open Riemann surface S, then there is $g' \varepsilon H(S)$ such that $g'(a) = 0$ and $g'(w) \neq 0$ for $w \neq a$, Behnke and Sommer (1962). Set $g' \,|Y = g$. Then $g(a) = 0$ and $g(w) \neq 0$ when $a \neq w$. $\Phi(g) \varepsilon R_\varphi$. If $\Phi(g)\,(z) = g(\varphi(z)) \neq 0$ for $z \varepsilon X$, then there is $h \varepsilon R_\varphi$ such that $\Phi(g)\,h = 1$. $h \varepsilon \Phi(H(Y))$ implies $h = \Phi(k)$, $k \varepsilon H(Y)$. $\Phi(g)\ \Phi(k) = 1$ implies $\Phi(gk) = 1$ but Φ is an isomorphism, thus $gk = 1$ and $g(a)\,k(a) = 1$. This contradicts $g(a) = 0$. Therefore $g(\varphi(z)) = 0$ for some $z \varepsilon X$ and $\varphi(z) = a$ for some $z \varepsilon X$. Thus φ is a mapping of X onto Y.

If R_φ has property (β^*) and $R_\varphi \neq C$, the same result holds.

Theorem 3. R_φ has property (Υ^*) if φ is a one-to-one mapping of X into Y.

Proof. Let $f \varepsilon R_\varphi$ and is f not identically zero. $f = \Phi(h)$ for some $h \varepsilon H(Y)$. h is not the constant 0 or else $f = \Phi(h) = 0$. Since H(Y) has property (Υ^*), there are functions $h_1,\ldots,h_n$ in H(Y) such that if $z \neq w$ and $h(z) = h(w) = 0$, then there is h_i such that $h_i(z) \neq h_i(w)$. $\{\Phi(h_i)\colon 1 \leq i \leq n\} \subseteq R_\varphi$. Suppose $x \neq y$ and $f(x) = f(y) = 0$. Then $h(\varphi(x)) = h(\varphi(y)) = 0$ and φ is one-to-one implies $\varphi(x) \neq \varphi(y)$, so there is a function h_i, $i=1,\ldots,n$, such that $h_i(\varphi(x)) \neq h_i(\varphi(y))$ or $\Phi(h_i(x)) \neq \Phi(h_i(y))$.

Theorem 4. R_φ separates the points of X *if* and only *if* φ is one-to-one.

Proof. Suppose R_φ separates the points of X. Let x and y belong to X, $x \neq y$. Then there is $f \varepsilon R_\varphi$ such that $f(x) \neq f(y)$. $f = \Phi(g) = go\varphi$ implies $g(\varphi(x)) \neq g(\varphi(y))$. If $\varphi(x) = \varphi(y)$, then $g(\varphi(x)) = g(\varphi(y))$ for every $g \varepsilon H(Y)$. Thus $\varphi(x) \neq \varphi(y)$ and φ is a one-to-one mapping of X into Y.

Now suppose φ is a one-to-one mapping of X into Y. Let x,y belong to X,x $\neq$ y. Then $\varphi(x)$, $\varphi(y)$ belong to Y and $\varphi(x) \neq \varphi(y)$.H(Y) separates the points of Y implies there is g ε H(Y) such that $g(\varphi(x)) \neq g(\varphi(y))$ which implies $\Phi(g(x)) \neq \Phi(g(y))$. $\Phi(g)$ ε R_φ. Thus R_φ separates the points of X.

Theorem 5. If R_φ properly contains C and has property (Υ^*), then φ is a one-to-one mapping of X into Y.

Proof. By Theorem 3, R_φ separates the points of X if and only if φ is a one-to-one function. We shall show $R_\varphi \neq$ C and R_φ has property (Υ^*) implies R_φ separates the points of X. Let x,y belong to X, x $\neq$ y. If there is f ε $R\varphi$, f is not identically zero, such that $f(x) = f(y) = 0$, then there is a set of function $\{f_1, \ldots, f_n\}$ in R_φ and a function f_i in the set such that $f_i(x) \neq f_i(y)$ by property (Υ^*). Since $R_\varphi \neq$ C there is g ε R_φ such that g is not a constant function. If $g(x) \neq g(y)$, then g separates x and y. If $g(x) = g(y) = c$, then since $c(x) = c$ belong to R_φ, g—c belongs to R_φ and $(g—c)(x) = (g—c)(y) = 0$. g $\not\equiv$ c since g is not a constant function. Since R_φ has property (Υ^*) there is h ε R_φ such that $h(x) \neq h(y)$. R_φ separates the points of X implies φ is one-to-one.

We make the following

Observations

(1) If $R\varphi \neq$ C and has property (α^*), then φ is an onto mapping from X to Y. If R_φ is to be a proper subring of H(X), then φ may not be a one-to-one mapping also. This implies R_φ does not separate the points of X.

(2) If $R_\varphi \neq C, R_\varphi$ can not have both properties (α^*) and (Υ^*) because then φ would be one-to-one and onto and $R_\varphi = H(X)$.

(3) R_φ separates the points of X implies φ is one-to-one. If R_φ properly contains C and is not H(X), then φ may not be an onto mapping and R_φ does not have property (α^*).

(4) By Theorems 3,4, and 5, if $R_\varphi \neq$ C the statements: R_φ has property (Υ^*), R_φ separates the points of X, φ is a oneto-one mapping of X into Y are equivalent.

(5) R_φ can not be an ideal of H(X) because 1 ε R_φ.

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