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RELATIONS BETWEEN THE MEAN CURVATURES OF THE PARALLEL SUBMANIFOLDS

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ABSTRACT

In this paper, we give a formulae for ι -th mean curvature of parallel submanifold to a given submanifold in E^m in the terms of principal curvatures of the given submanifold. In addition we give a formulae for ι - th mean curvatures of paralel submanifold in E^m in the terms of ι - th mean curvatures of the given submanifold.

Finally, we examine these formulaes for parallel surfaces in E^3 and parallel hypersurfaces in E^m.

INTRODUCTION

We will remind some basic properties of submanifolds and parallel submanifolds.

Let N and M be m and n-dimensional Riemannian manifolds, respectively, on the condition that N is an immersed submanifold into M. Let us denote the immersion by f. If there is no confusion we identify the manifolds M and f(M) and the points x and f(x). Thus, the tangent space T_xM of M at the point x is a vector subspace of T_xN of the submanifold N at the point x. We denote the normal bundle of M by $T^{\perp}(M)$ and the covariant derivative on $T^{\perp}(M)$ by D. If we denote by ∇ and ∇' Riemannian connections on M and N, respectively, and the second fundamental form of M by α. Then, we have

$$\nabla'_X Y = \nabla_X Y + \alpha(X,Y)$$

and

$$\nabla'_X \zeta = - A\zeta\ (X) + D_X \zeta$$

for any two vector field X,Y tangent to M and any vector field ζ normal to M, where A_ζ denotes the Weingarten map with respect to ζ (Kobayashi and Nomizu, 1969).

Definition 1.1. M be an n-dimensional immersed submanifold of a Riemannian manifold N and η a unit normal vector to M at a point p. Let $k_1, k_2, \ldots, k_n$ be principal curvatures of M with respect to η. we put

$$\binom{n}{\iota} M_\iota(\eta) = \sum_{1 \leq i_l < \cdots < i_l \leq n} k_{i_1} k_{i_2} \ldots k_{i_l}, \quad M_0(\eta) = 1,$$

where $\binom{n}{\iota} = n!/(n-\iota)!\iota!$. We call $M_l(\eta)$ the ι-th mean curvature with respect to η (Chen, 1973).

Definition 1.2. Let M be a C^∞ n-dimensional regular submanifold of E^m such that M is regular and arcwise connected Let ζ be a unit normal section of M and r be a real number such that $1/r$ is not equal to any principal curvatures of M at any point of M in the sense that $1/r$ is not an eigen value of A_ζ, the shape operator of M with respect to ζ. Define a function

$$f : M \rightarrow E^m$$
$$P \rightarrow f(P) = P + r\zeta_p$$

Then f(M) is called parallel submanifold of M with respect to ζ if it is endowed with the C^∞ structure induced by f from M. If f(M) is a parallel submanifold of M with respect to ζ then we will write $M_{\zeta,r}$ as f(M) and we will understand $M//M_{\zeta,r}$ for $M_{\zeta,r}$ is parallel submanifold of M (Görgülü and Özdamar, 1989).

Theorem 1.1. Let $M//M_{\zeta,r}$. Then

$$f_*(v_p) = v_p + r(-A_\zeta (v_p) + D_{v_p}\zeta)$$

for every $v_p \in T_pM$ (Görgülü and Özdamar, 1989).

Theorem 1.2. If $M//M_{\zeta,r}$ then ζ is a unit normal section of $M_{\zeta,r}$ by identifying $\zeta_{f(p)}$ with ζ_p, that is, ζ_p is translated to f(p) by Euclidean parallelism in E^m (Görgülü and Özdamar, 1989).

Theorem 1.3. Let $M//M_{\zeta,r}$. Let ζ and $\overline{\zeta}$ be unit normal parallel sections on M and $M_{\zeta,r}$, respectively, and $\zeta(p) = \overline{\zeta}(f(p))$. Then

$$A_{\zeta,r}(f_*(v_p)) = A_\zeta(v_p)$$

for every $v_p \varepsilon T_pM$, where A_ζ and $A_{\zeta,r}$ denote the shape operators of M and $M_{\zeta,r}$ with respect to ζ, respectively (Görgülü and Özdamar, 1989).

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Theorem 2.1. Let $M//M_{\zeta,r}$. Let ζ be a unit normal parallel section on M Then f preserves principal directions with respect to ζ.

Proof: Let v_p be a principal direction with respect to ζ and denote the principal curvature by k corresponding to v_p, Then we have

$$A_\zeta(v_p) = kv_p.$$

Since ζ is parallel in the normal bundle we can write the following

$$A_{\zeta,r}(f_*(v_p)) = A_\zeta(v_p)$$

Thus

$$\begin{aligned} f_*(v_p) &= v_p - rA_\zeta(v_p) \\ &= (1-rk)\, v_p \end{aligned}$$

or

$$v_p = (1/(1-rk))\, f_*(v_p)$$

Hence

$$\begin{aligned} A_{\zeta,r}(f_*(v_p)) &= A_\zeta(v_p) \\ &= k\, v_p \\ &= \frac{k}{1-rk}\, f_*(v_p) \end{aligned}$$

which completes the proof.

Corollary 2.1. Let $M//M_{\zeta,r}$. Let ζ be a parallel normal section of M. If the submanifold M is umbilical with respect to ζ, then $M_{\zeta,r}$ is also umbilical with respect to $\bar{\zeta} = \zeta_r$.

Proof: One can easly show that since it is similar to the proof of the theorem 2.1.

Theorem 2.2. Let ζ be a parallel unit normal vector field. Denote $M_l^r(\zeta_r)$, $0 \leq \iota \leq n$, for ι-th mean curvatures of $M_{\zeta,r}$ with respect to ζ_r. Then

$$M_l^r(\zeta_r) = \frac{1}{\binom{n}{l}} \sum_{1 \leq i_1 < \dots < i_l \leq n} \prod_{s=1}^{\iota} \frac{k_{i_s}}{1-rk_{i_s}}$$

where k_i, $1 \leq i \leq n$, denote principal curvatures of M with respect to ζ.

Proof: Let $E_1, \dots, E_n$ be the principal directions of the normal section ζ with the principal curvatures. So

$$A_\zeta(E_i) = k_i E_i \ , \ 1 \leq i \leq n$$

Since ζ is parallel in the normal bundle then we have

$$\begin{aligned} A_{\zeta,r}(f_*(E_i)) &= A_\zeta(E_i) \\ &= k_i E_i \end{aligned}$$

On the other hand, since

$$f_*(E_i) = (1-rk_i)\ E_i$$

or

$$E_i = \frac{1}{1-rk_i} f_*(E_i)$$

We have the following

$$A_{\zeta,r}(f_*(E_i)) = \frac{k_i}{1-rk_i} f_*(E_i)$$

Thus, we get

$$M_l^r(\zeta_r) = \frac{1}{\binom{n}{l}} \sum_{1 \leq i_1 < \dots < i_l \leq n} \frac{k_{i_1}}{1-rk_{i_1}} \cdots \frac{k_{i_\iota}}{1-rk_{i_l}}$$

$$M_l^r(\zeta_r) = \frac{1}{\binom{n}{l}} \sum_{1 \leq i_1 < \dots < i_l \leq n} \prod_{s=1}^{\iota} \frac{k_{i_s}}{1-rk_{i_s}}$$

as desired.

Theorem 2.3. Let $M//M_{\zeta,r}$. Let ζ be a unit normal parallel vector field. Then

$$M_\iota^r(\zeta_r) = \frac{\sum_{s=\iota}^{n} (-1)^{s-\iota} r^{s-\iota} \binom{n}{s} \binom{s}{s-\iota} M_s(\zeta)}{\binom{n}{\iota} \sum_{k=0}^{n} (-1)^k \binom{n}{k} r^k M_k(\zeta)}, \quad 1 \leq \iota \leq n,$$

where $M_\iota(\zeta)$ and $M_\iota^r(\zeta_r)$ denote ι-th mean curvatures of M and $M_{\zeta,r}$ with respect to ζ.

Proof: One can easily show that

$$\sum_{1\leq i_1<\ldots<i_\iota\leq n} \prod_{s=1}^{\iota} \frac{k_{i_s}}{1-rk_{i_s}} = \frac{\sum_{s=\iota}^{n} (-1)^{s-\iota}\binom{s}{s-\iota} r^{s-\iota} \sum_{1\leq i_1<\ldots<i_s\leq n} k_{i_1}\ldots k_{i_s}}{\prod_{i=1}^{n} (1-rk_i)}$$

and

$$\prod_{i=1}^{n} (1-rk_i) = 1 + \sum_{s=1}^{n} (-1)^s r^s \sum_{1\leq i_1<\ldots<i_s\leq n} k_{i_1}\ldots k_{i_s}$$

Since

$$M_0(\zeta) = 1 \text{ and } \binom{n}{s} M_s(\zeta) \sum_{1\leq i_1<\ldots<i_s\leq n} k_{i_1}\ldots k_{i_s}$$

so we have that the following formulaes

$$M_\iota^r(\zeta_r) = \frac{1}{\binom{n}{\iota}} \sum_{1\leq i_1<\ldots<i_\iota\leq n} \prod_{s=1}^{\iota} \frac{k_{i_s}}{1-rk_{i_s}}$$

$$= \frac{\sum_{s=\iota}^{n} (-1)^{s-\iota} r^{s-\iota} \binom{n}{s} \binom{s}{s-\iota} M_s(\zeta)}{\binom{n}{\iota} \sum_{k=0}^{n} (-1)^k \binom{n}{k} r^k M_k(\zeta)}$$

SPECIAL CASES:

1. In the special case of $n = 2$ and $m < 3$, M and $M_{\zeta,r}$ become 2–dimensional parallel submanifolds of E^m. In that case there exist just 1–th mean curvature $M^r_1(\zeta_r)$ and 2–th mean curvature $M^r_2(\zeta_r)$ of $M_{\zeta,r}$. Thus

$$M^r_1(\zeta_r) = \frac{2\, M_1(\zeta) - 2\, rM_2(\zeta)}{2(1-2\, rM_1(\zeta) + r^2 M_2(\zeta))}$$

and

$$M^r_2(\zeta_r) = \frac{M_2(\zeta)}{1-2\, rM_1(\zeta) + r^2 M_2(\zeta)}$$

2. In the case of $n = 2$ and $m = 3$, M and $M_{\zeta,r}$ become parallel surfaces of E^3. Since

$$M_1(\zeta) = \frac{1}{2} H$$

$$= \frac{1}{2}(k_1 + k_2)\ ,$$

$$M_2(\zeta) = K$$

$$= k_1 k_2$$

and

$$M^r_1(\zeta_r) = \frac{1}{2} H_r$$

$$= \frac{1}{2}\left(\frac{k_1}{1-rk_1} + \frac{k_2}{1-rk_2}\right)$$

$$M^r_2(\zeta_r) = K_r$$

$$= \frac{k_1}{1-rk_1} \cdot \frac{k_2}{1-rk_2}$$

Notice that we have in that case the following formulaes

$$H_r = \frac{H - 2r\, K}{1-rH + r^2\, K}$$

$$K_r = \frac{K}{1 - rH + r^2 K}$$

which are the same as in (Hacısalihoğlu, 1983), The Theorem 4, 7, 3.

3. In the case of $n = n$ and $m = n + 1$, we get all the relations between the higher curvatures of the parallel hypersurfaces in (Görgülü, 1985).

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