Commun Fac. Sci. Univ. Ank. Series A_1
V. 41. pp. 67-71 (1992)

THE FACTORIZATION OF ELEMENTS OF SO(n) IN TERMS OF THE EULERS ANGLES

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(Received Nov. 6, 1991; Accepted July 14, 1992)

SUMMARY

In [1], the orthogonal matrices have been obtained with the aid of Skew-symmetric matrices in E^2 and E^3, In addition, an interpretation for these matrices has been given in this paper we tried to give the solution of this problem for E^n, $n > 3$.

It has been shown in [2] that the Lie algebra of the Lie group of orthogonal matrices O(n) consists of the skew symmetric matrices. Here, we explain the orthogonal nxn matrices in terms of the exponential expansion of the bases of skew-symmetric matrices. Consequently we give a factorization of elements of SO(n) in terms of Euler Angles.

INTRODUCTION

Lie Groups And Lie Algebras

Definition 1.1. (Lie Groups). A Lie group is a group G which is, at the same time, a differentiable manifold such that the group operation

$$\square: G \times G \rightarrow G$$
$$(a, b) \rightarrow ab^{-1}$$

is a differentiable mapping of G x G (product manifold) into G.

Definition 1.2. (Lie Algebra). We define the Lie algebra of a Lie group G as the Lie algebra of the left invariant vector fields on G. We have

$$\chi_l(G) \cong T_G(e)$$

where $\chi_l(G)$ is the space of left invariant vector fields on G and e is the identity element of G.

Theorem 1.1. The Lie algebra of the Lie group O(n) is the space of nxn skew-symmetric matrices [2].

Proof. Let $g \in O(n)$ then we have

$$g^T.g = e$$

$$d\,(g^T.g) = 0$$

$$(g^{-1}dg)^T + g^{-1}\,dg = 0.$$

Hence, $\omega_{ij}|_g = g^{-1}_{ik}dg_{kj} \in T^*_{GL(n,IR)}(g)$ we obtain

$$[\omega_{ij}|_g]^T + [\omega_{ij}|_g] = 0$$

or

$$\omega_{ji}|_g + \omega_{ji}|_g = 0.$$

For the inclusion mapping

$$i^*: T^*_{GL(n,IR)}(g) \to T^*_{O(n)}(g)$$

we have

$$i^*(\omega_{ji}|_g) + i^*(\omega_{ij}|_g) = 0$$

$$\xi_{ji}|_g + \xi_{ij}|_g = 0 \Rightarrow \xi_{ij} = -\xi_{ji},\ i^*(\omega_{ji}|_g) = \xi_{ji}|_g,$$

this proves the theorem.

If we denote the space of left invariant forms on O(n), by $\Omega_l(O)$, then a base for this space is $\{\xi_{ij}\}$. Since a dual of this base is also a base for $T_{O(n)}e$, then a base of $T_{O(n)}e$ is

$$\left\{ \frac{\partial}{\partial x_{ij}} - \frac{\partial}{\partial x_{ij}},\ 1 \leq i, j \leq n \right\}$$

$T_{O(n)}(e)$, which is the Lie algebra of O(n), is the space of n x n skew-symmetric matrices.

2. SO(n) And The Angles Of Euler

Theorem 2.1. If L is an n x n skew-symmetric matrix then $e^{L\theta} \in SO(n)$.

Proof. Let $A = e^{L\theta}$

$$\begin{aligned} A.A^T &= (e^{L\theta})\,(e^{L\theta})^T \\ &= e^{L\theta}\,e^{L^T\theta} \qquad L^T = -L \\ &= e^{L\theta - L\theta} \\ &= e^{o} \\ &= I_n. \end{aligned}$$

Since we have

$$A^T A = I_n$$

then

$$A \in O(n) \quad \dots\dots\dots\dots\dots\dots \quad (1)$$

Since we have [3]

$$\det e^L = e^{izL}$$

$$L = [l_{ij}] \ , \quad iz\, L = 0$$

$$\det e^L = e^o$$

$$\det e^L = 1 \quad \dots\dots\dots\dots\dots \quad (2)$$

So, (1) and (2) give us that $A \in SO(n)$.

Hence we can say that the Lie algebra of O(n) is the space of skew-symmetric matrices. Let $\{L_1, L_2, \dots, L_{\frac{n(n-1)}{2}}\}$ be a base of this space, then the elements of this base can be written as

$$L_1 = \begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0 & 0 & -1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\vdots$$

$$L_{\frac{n(n-1)}{2}} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Theorem 2.1. tells us that all the matrices $e^{L_i\theta_i}$, $1 \leq i \leq \frac{n(n-1)}{2}$, are the rotation matrices. Each of these matrices represents a rotation about an axis. Hence we may consider here a composition of these $\frac{n(n-1)}{2}$ rotations.

$e^{L_1\theta_1}$ causes a rotation about the axis $\frac{\partial}{\partial x_1}$ by the angle θ_1,

$e^{L_2\theta_2}$ causes a rotation about the axis $\frac{\partial}{\partial x_2}$ by the angle θ_2,

$e^{L_{\frac{n(n-1)}{2}}\theta_{\frac{n(n-1)}{2}}}$ causes a rotation about the axis $\frac{\partial}{\partial x_{\frac{n(n-1)}{2}}}$

by the angle $\theta_{\frac{n(n-1)}{2}}$.

Further, if we have the product of these orthogonal matrices we obtain the matrix A such as

$$A = e^{L_{\frac{n(n-1)}{2}}\theta_{\frac{n(n-1)}{2}}} \cdots e^{L_1\theta_1}. \qquad (*)$$

Then A is also an orthogonal matrix since the orthogonal matrices form a group under the matrix multiplication. Moreover, since

$$\det A = 1$$

we have that $A \in SO(n)$. When we consider the angles θ_i as Euler angles in (*), then the matrix $A \in SO(n)$ has a factorization in terms of the Eular angles.

3. An Example for the Case of n = 3

The Lie algebra of Lie group O(3) consists of the matrices in the form

$$L = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

each of which is the skew–symmetric so we can write this matrix as

$$L = a\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Denoting

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, L_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

we have

$$e^{L_1\theta_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}, e^{L_2\theta_2} = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix},$$

$$e^{L_3\theta_3} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For these matrices,

$e^{L_1\theta_1}$ causes a rotation about $\dfrac{\partial}{\partial x_1}$ by the angle θ_1,

$e^{L_2\theta_2}$ causes a rotation about $\dfrac{\partial}{\partial x_2}$ by the angle θ_2,

$e^{L_3\theta_3}$ causes a rotation about $\dfrac{\partial}{\partial x_3}$ by the angle θ_3.

In addition, if we have the product of these matrices we obtain

$$A = e^{L_3\theta_3} \cdot e^{L_2\theta_2} \cdot e^{L_1\theta_1}$$

then we have that $A \in SO(3)$.

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